Technische Universität München
Zentrum Mathematik

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Winter term 2017/18
Exercise sheet 6

Linear algebraic groups (MA 5113)

Exercise 21 (Product of projective spaces)**.** Let $m, n \in \mathbb{N}$. Define a morphism of varieties

$$\phi \colon \mathbb{P}^m \times \mathbb{P}^n \to \mathbb{P}^{mn+m+n}, ((x_i),(y_j)) \mapsto (x_i y_j).$$

Prove that ϕ is a closed embedding (i.e. the image $V_{m,n}$ of ϕ is closed and ϕ defines an isomorphism of $\mathbb{P}^m \times \mathbb{P}^n$ to $V_{m,n}$). Deduce that the product of two (quasi-)projective varieties is again (quasi-) projective.

Exercise 22 (Example of a quotient)**.** Let $B \subset \mathrm{GL}_2$ be the subgroup of all upper triangular matrices. Prove that

$$\mathrm{GL}_2 / B \cong \mathbb{P}^1$$

where the GL_2-action on $\mathbb{P}^1$ is the Möbius action described in Exercise 10.

Exercise 23 (Irreducibility of quotients)**.** The following are equivalent

(a) G/H is connected.
(b) G/H is irreducible.
(c) H meets all connected components of G.
(d) $G = G^0 H$

Exercise 24 (Quasi-affine quotients)**.** Let $H \subset G$ be a normal subgroup

(a) With V, v as in Corollary 2.6.9, prove that there exists a morphism of algebraic groups $\rho' \colon H \to \mathbb{G}_m$ such that for any $h \in H, w \in W = \langle v \rangle$ we have $\rho(h)w = \rho'(h) \cdot w$, where we consider the scalar multiplication of $k^\times$ on W on the right hand side.
(b) Assume that the only morphism of linear algebraic groups $H \to \mathbb{G}_m$ is the trivial morphism mapping every element to 1. Show that G/H is quasi-affine, i.e. isomorphic to an open subvariety of an affine variety.
(c) Find a non-trivial example $H \subset G$, where the prerequisite of (b) is satisfied.

Deadline: Friday, December 1, 2017

If you have questions regarding the exercises, please send an email to hamacher@ma.tum.de.
The exercise classes are Fridays, 10-12 in room MI 02.08.020.
Further information about our lectures and exercises are available under

`http://www-m11.ma.tum.de/viehmann/viehmann-linear-algebraic-groups/`.