Technische Universität München
Zentrum Mathematik

Prof. Dr. Eva Viehmann
Dr. Paul Hamacher

Winter term 2017/18
Exercise sheet 7

Linear algebraic groups (MA 5113)

Exercise 25 (Abelian subgroups of a linear algebraic group)**.** Let G be a linear algebraic group.

(a) Prove that if $H \subset G$ is any Abelian subgroup, then $\overline{H}$ is also Abelian.
(b) Conclude that any element $g \in G$ is contained in a closed Abelian subgroup. Moreover, if g is semisimple or unipotent, this subgroup can be chosen to have the same property.

Exercise 26 (Subgroups of diagonalizable groups)**.** Let G be a diagonalizable linear algebraic group and $H \subset G$ a closed connected subgroup. Prove that there exists a closed subgroup $H' \subset G$ such that the multiplication defines an isomorphism $H \times H' \xrightarrow{\sim} G$. Also give a counterexample which proves that we cannot drop the assumption that G is connected.

Exercise 27 (Character of an arbitrary linear algebraic group)**.** Let G be a linear algebraic group and let

$$H := \bigcap_{\chi \in X^*(G)} \ker \chi.$$

Prove that
(a) H is a closed normal subgroup of G.
(b) G/H is diagonalizable.
(c) $X^*(G) = X^*(G/H)$.
In particular, $X^*(G)$ is finitely generated Abelian group. Determine $X^*(\mathrm{GL}_n)$

Exercise 28 (Representations of $\mathbb{G}_m$)**.** Let $\mathcal{C}$ be the category of pairs (ρ, V) where $\rho : \mathbb{G}_m \to V$ is a representation, and where morphisms $(\rho, V) \to (\rho', V')$ are linear maps $f : V \to V'$ with $f(\rho(g)v) = \rho'(g)f(v)$ for all $g \in \mathbb{G}_m$ and $v \in V$.
Let $\mathcal{D}$ be the category of $\mathbb{Z}$-graded vector spaces, i.e. of vector spaces V with a decomposition $V = \bigoplus_i V_i$, and where morphisms are linear maps $f : V \to V'$ with $f(V_i) \subseteq V'_i$ for all i.
Prove that $\mathcal{C}$ and $\mathcal{D}$ are equivalent.

Deadline: Friday, December 8, 2017

If you have questions regarding the exercises, please send an email to hamacher@ma.tum.de.
The exercise classes are Fridays, 10-12 in room MI 02.08.020.
Further information about our lectures and exercises are available under

`http://www-m11.ma.tum.de/viehmann/viehmann-linear-algebraic-groups/`.