Technische Universität München
Zentrum Mathematik

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Winter term 2017/18
Exercise sheet 11

# Linear algebraic groups (MA 5113)

**Exercise 41** (Reflections and the Weyl group)**.** Let $G$ be a linear algebraic group with maximal torus $T$ and Weyl group $W$. Let $P_0 \subset X_*(T)$ denote the set of weights for the adjoint action of $T$ on $\operatorname{Lie} G$ for which the group $G_\alpha$ is non solvable.

(a) Prove that $P_0$ is stable under the $W$-action on $X^*(T)$.

(b) Show that for every $\alpha \in P_0, w \in W$ we have $s_{w(\alpha)} = w s_\alpha w^{-1}$

*Classical linear algebraic groups:* We consider the symplectic group $\mathrm{Sp}_{2n}$ and the special orthogonal group $\mathrm{SO}_n$. We briefly recall their definition. Let $J \coloneqq (\delta_{i,n+1-j})_{i,j} \in \mathrm{GL}_n$ be the matrix with entries 1 on the antidiagonal and 0 otherwise and let $J_a \coloneqq \left(\begin{smallmatrix} 0 & -J \\ J & 0 \end{smallmatrix}\right) \in \mathrm{GL}_{2n}$. Then we may define

$$\mathrm{SO}_n \coloneqq \{g \in \mathrm{SL}_n \mid g^T \cdot J \cdot g = J\}$$
$$\mathrm{Sp}_{2n} \coloneqq \{g \in \mathrm{GL}_{2n} \mid g^T \cdot J_a \cdot g = J_a\},$$

see also Exercise 36(a) (a similar argument can also be used for symplectic forms). It is known that $\mathrm{SO}_n$ and $\mathrm{Sp}_n$ are connected and that their Lie algebras are given by

$$\mathfrak{so}_n = \{A \in k^{n\times n} \mid A^T \cdot J + J \cdot A = 0\}$$
$$\mathfrak{sp}_{2n} = \{A \in k^{2n\times 2n} \mid A^T \cdot J_a + J_a \cdot A = 0\},$$

see Exercise 19.

**Exercise 42** (Weyl group of $\mathrm{SO}_n$)**.** Let $T \subset \mathrm{SO}_n$ be the subgroup of diagonal elements.

(a) Prove that $T$ is a maximal torus (*Hint:* Exercise 36(b) ) and give an explicit desciption of $X^*(T)$ and $X_*(T)$.

(b) Determine the set $P \subset X^*(T)$ of weights for the adjoint action of $T$ on $\mathfrak{so}_n$.

(c) Calculate the Weyl group of $\mathrm{SO}_n$ (*Hint:* Theorem 4.1.10)

**Exercise 43** (Weyl group of $\mathrm{Sp}_{2n}$)**.** Let $T \subset \mathrm{Sp}_{2n}$ be the subgroup of diagonal elements. You can use without proof that $T$ is a maximal torus.

(a) Give an explicit desciption of $X^*(T)$ and $X_*(T)$.

(b) Determine $X^*(T)$ and the set $P \subset X^*(T)$ of weights for the adjoint action of $T$ on $\mathfrak{sp}_{2n}$.

(c) Calculate the Weyl group of $\mathrm{Sp}_{2n}$.

**Exercise 44** (Groups of dimension 2)**.** Let $G$ be a connected linear algebraic group of dimension 2. Prove that $G$ is solvable.

Deadline: Friday, 21st January, 2017

**If you have questions regarding the exercises, please send an email to hamacher@ma.tum.de.**
**The exercise classes are Fridays, 10-12 in room MI 02.08.020.**
**Further information about our lectures and exercises are available under**

`http://www-m11.ma.tum.de/viehmann/viehmann-linear-algebraic-groups/`.