Technische Universität München
Zentrum Mathematik

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Exercise sheet 12

Linear algebraic groups (MA 5113)

In Exercise 45 you may use the following result from algebraic geometry without proof.

Theorem. *Let $f\colon X \to Y$ be a morphism of varieties such that for every $y \in Y$ the fibre* $\dim f^{-1}(y)$ *das dimension $d \in \mathbb{N}$ (in particular f is surjective). Then* $\dim X = \dim Y + d$.

Exercise 45 (Extensions of linear algebraic groups)**.** Let $\phi\colon G \to H$ be a morphism of linear algebraic group.

(a) Show that $\dim G = \dim \ker \phi + \dim \operatorname{im} \phi$

(b) Assume that the induced morphism $G \to \operatorname{im} \phi$ is separable. Prove that

$$\ker d\phi_e = T_e(\ker \phi)$$
$$\operatorname{im} d\phi_e = T_e(\operatorname{im} \phi)$$

Exercise 46 (Central extensions and root data)**.** Let $\phi\colon G \to G'$ be a surjective, separable morphism of linear algebraic groups such that $Z := \ker \phi$ is diagonalisable and is contained in the center of G. Let T be a maximal torus of G, then $T' := \phi(T)$ is a maximal torus of G' (see Exercise 40). Denote by $(X, R, X^\vee, R^\vee)$ the root datum of G and by $(X', R', X'^\vee, R'^\vee)$ the root datum of G'. Let $\phi_X : X' \to X$ and $\phi_{X^\vee} : X^\vee \to X$ be the homomorphisms defined by composition with ϕ.

(a) Show that $\phi_{X^\vee}$ is the dual of ϕ_X with respect to $\langle\ ,\ \rangle$.

(b) Prove that ϕ_X and $\phi_{X^\vee}$ induce bijections between R and R', and $R^\vee$ and $R'^\vee$ respectively.

Remark: In other words, we can identify the roots (and coroots, respectively) of G and G' preserving the pairing $\langle\ ,\ \rangle$; only the surrounding $\mathbb{Z}$-modules change. One says that the root data have the same type.

Exercise 47 (Geometry of the Weyl group)**.** Let $(X, R, X^\vee, R^\vee)$ be a root datum and $(\ ,\ )$ be a W-invariant bilinear form on $V := X \otimes \mathbb{R}$. Recall that $s_\alpha(x) = x - 2\frac{(x,\alpha)}{(\alpha,\alpha)} \cdot \alpha$ (see script, right before Lemma 4.2.5).

(a) Show that $W \subset O(V)$.

(b) Given two roots $\alpha, \beta \in R$, show that their angle is either $0°, 30°, 60°, 90°, 120°, 150°$ or $180°$.

(c) Assume we have chosen a set of positive roots $R^+ \subset R$ as in Definition 4.3.5. Prove that for any two linearly independent roots $\alpha, \beta \in R$, there exists a $w \in W$ such that $w(\alpha) > 0, w(\beta) > 0$.

Exercise 48 (Classification of root systems of rank 2)**.** Let $(X, R, X^\vee, R^\vee)$ be a root datum and let $\alpha, \beta \in R$ linearly independent. By Lemma 4.4.1, there exists an $a \in \{0, 1, 2, 3\}$ such that if after interchanging α and β, and/or replacing α by $-\alpha$, we may assume that $\langle \alpha, \beta^\vee \rangle = -a$, $\langle \beta, \alpha^\vee \rangle = -1$ if $a \neq 0$ and $\langle \alpha, \beta^\vee \rangle = \langle \beta, \alpha^\vee \rangle = 0$ if $a = 0$. Prove that the set S descibed below is contained in R and stable under s_α and s_β.
(a) If $a = 0$, $S = \{\pm\alpha, \pm\beta\}$,
(b) if $a = 1$, $S = \{\pm\alpha, \pm\beta, \pm(\alpha + \beta)\}$,
(c) if $a = 2$, $S = \{\pm\alpha, \pm\beta, \pm(\alpha + \beta), \pm(2\alpha + \beta)\}$ and
(d) if $a = 3$, $S = \{\pm\alpha, \pm\beta, \pm(\alpha + \beta), \pm(2\alpha + \beta), \pm(3\alpha + \beta), \pm(3\alpha + 2\beta)\}$.
Remark: Note that S forms a root system. We will later see that if $R \subset \mathbb{Z}\alpha + \mathbb{Z}\beta$, then we must have $R = S$. Thus the list above contains all root systems of rank 2.

Deadline: Friday, 28th January, 2017

If you have questions regarding the exercises, please send an email to hamacher@ma.tum.de.
The exercise classes are Fridays, 10-12 in room MI 02.08.020.
Further information about our lectures and exercises are available under

`http://www-m11.ma.tum.de/viehmann/viehmann-linear-algebraic-groups/`.