Technische Universität München
Zentrum Mathematik

Prof. Dr. Eva Viehmann
Dr. Paul Hamacher

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Exercise sheet 13

Linear algebraic groups (MA 5113)

Exercise 49 (The symmetric group as Coxeter group)**.** Let $G = \mathrm{GL}_n$ and denote by $T \subset B \subset G$ the diagonal torus and the Borel subgroup of upper triangular matrices, respectively. Then the Weyl group of (G, T) is the symmetric group $\mathfrak{S}_n$ by Exercise 10.

(a) Let $S \subset \mathfrak{S}_n$ be the set of simple reflections corresponding to B. Prove that S equals the set of adjacent transpositions $\{(i \;\; i+1) \mid i = 1, \dots n-1\}$.

(b) Let $w \in \mathfrak{S}_n$. Prove that $\ell(w) = \#\{(i,j) \in \{1, \dots, n\}^2 \mid i < j, \sigma(i) > \sigma(j)\}$

Exercise 50 (The Weyl group of Sp_{2n})**.** Let $G = \mathrm{Sp}_{2n}$ and denote by $T \subset B \subset G$ the diagonal torus and the Borel subgroup of upper triangular matrices, respectively. Then the Weyl group W of (G, T) is isomorphic to $\{w \in \mathfrak{S}_n \mid w(2n+1-i) = 2n+1-w(i)\}$ by Exercise 43.

(a) Show that W is isomorphic to the group of automorphisms of $\{\pm 1, \dots, \pm n\}$ satisfying $\sigma(-a) = -\sigma(a)$ for all a.

(b) Let $S \subset W$ be the set of simple reflections corresponding to B. Prove that

$$S = \{(i \;\; i+1)(2n-i \;\; 2n+1-i) \mid i = 1, \dots, n-1\} \cup \{(n \;\; n+1)\}.$$

(c) For each pair of simple reflections s_i, s_j compute the order of $s_i s_j$.

Exercise 51 (Inner and outer automorphisms)**.** Let G be semi-simple reductive group. We fix a *splitting* of G, that is a triple $(B, T, \{u_\alpha\}_{\alpha \in D})$, where $T \subset B \subset G$ is a maximal torus and a Borel subgroup, respectively; $u_\alpha \in U_\alpha$ where we denote by D the basis of the root system associated to B. Moreover, we denote by $\Phi = (X, R, D, X^\vee, R^\vee, D^\vee)$ the root datum with basis associated to (G, B, T).

(a) Show that G acts simply transitively on the set of splittings by conjugation. Conclude that any automorphism $\phi \in \mathrm{Aut}(G)$ can be written uniquely as $\phi = c_g \circ \phi_0$, where c_g denotes the conjugation with an element $g \in G$ (such an automorphism is called inner automorphism) and $\phi_0 \in \mathrm{Aut}\, G$ fixes $(B, T, \{u_\alpha\}_{\alpha \in D})$.

(b) Denote by $\mathrm{Out}(G)$ the set of automorphisms $\phi_0 \in \mathrm{Aut}(G)$ which fix $(B, T, \{u_\alpha\}_{\alpha \in D})$. Prove that any $\phi_0 \in \mathrm{Out}(G)$ induces an automorphism of Φ and that the resulting homomorphism $\mathrm{Out}(G) \to \mathrm{Aut}\, \Phi$ is injective.

(c) Show that for $G = \mathrm{SL}_n$, one has $\mathrm{Aut}\, \Phi \cong \mathbb{Z}/2\mathbb{Z}$. Give an example of an automorphism $\phi_0 \in \mathrm{Out}(SL_n)$ (for a splitting of your choice) such that it corresponds to the non-trivial automorphism in $\mathrm{Aut}\, \Phi$.

Exercise 52 (Bruhat decomposition for GL_n)**.** Let $G = \mathrm{GL}_n$ and $B \subset G$ the Borel subgroup of upper triangular matrices. Recall that G/B can be identified with the parameter space of flags $V_1 \subset \ldots \subset V_{n-1} \subset k^n$ with $\dim V_i = i$.

(a) Describe the canonical G-action on G/B in terms of flags.

(b) Prove that two flags $(W_\bullet), (W'_\bullet) \in G/B$ lie in the same B-orbit if and only if for every i, j

$$\dim W_i \cap V_j = \dim W'_i \cap V_j$$

(c) Give explicit conditions on the matrix entries describing the six B-double cosets for $n = 3$.

Deadline: Friday, 4th February, 2017

If you have questions regarding the exercises, please send an email to hamacher@ma.tum.de.
The exercise classes are Fridays, 10-12 in room MI 02.08.020.
Further information about our lectures and exercises are available under

`http://www-m11.ma.tum.de/viehmann/viehmann-linear-algebraic-groups/`.