Technische Universität München
Zentrum Mathematik

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Summer term 2017
Exercise sheet 2

Linear algebraic groups (MA 5113)

Exercise 5 (Principal open sets as affine varieties)**.** Let $(X, \mathcal{O}_X)$ be an affine variety over an algebraically closed field k and fix an element $f \in k[X]$. Let $(Y, \mathcal{O}_Y)$ be the affine variety corresponding to the reduced affine k-algebra $k[X]_f = \mathcal{O}_X(D(f))$. Prove that $(Y, \mathcal{O}_Y) \cong (D(f), \mathcal{O}_X|_{D(f)})$, in particular $(D(f), \mathcal{O}_X|_{D(f)})$ is an affine variety.

Exercise 6 (Properties of varieties)**.** The following properties distinguish the varieties from the prevarieties.

(a) Let $X \underset{g}{\overset{f}{\rightrightarrows}} Y$ be morphisms of prevarieties. Show that it the set $\{x \in X \mid f(x) = g(x)\}$, is closed in X if Y is a variety but that it is not necessarily closed if Y is only a prevariety.

(b) Let $f\colon X \to Y$ be a morphism of prevarieties. Show that the graph of f, that is the set $\Gamma_f \coloneqq \{(x, y) \in X \times Y \mid y = f(x)\}$ is closed in $X \times Y$ if Y is a variety but that it is not necessarily closed if Y is only a prevariety.

Exercise 7 (Centralizer and Normalizer)**.** Let G be an algebraic group and $H \subset G$ a subset.

(a) Show that the conjugation $G \times G \to G, (g, x) \mapsto gxg^{-1}$ is a morphism of varieties.

(b) Prove that the normalizer and centralizer of H in G, given by

$$N_G(H) \coloneqq \{g \in G \mid gHg^{-1} = H\}$$
$$Z_G(H) \coloneqq \{g \in G \mid \forall h \in H : gh = hg\}$$

are closed subgroups of G, where H is assumed to be a subgroup when considering its normalizer.

Exercise 8 (Normalizer of the diagonal torus)**.** Denote by $T \subset \mathrm{GL}_n$ the subgroup of diagonal matrices.

(a) Show that the normalizer $N_G(T)$ consists exactly of all generalised permutation matrices, i.e. the matrices which have exactly one nonzero entry in each row and each column.

(b) Determine the connected components of $N_G(T)$.

Deadline: Friday, 3rd November, 2017

If you have any questions regarding the exercises, please send an email to hamacher@ma.tum.de. The exercise classes are Fridays, 10-12 in room MI 02.08.020. Further information about our lectures and exercises are available under `http://www-m11.ma.tum.de/viehmann/viehmann-linear-algebraic-groups/`.