Technische Universität München
Zentrum Mathematik

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Exercise sheet 3

Linear algebraic groups (MA 5113)

Exercise 9 (The projective line)**.** We fix an algebraically closed field k and consider the equivalence relation on $k^2 \setminus \{0\}$ given by $v \sim w$ if there exists a $\lambda \in k^\times$ with $v = \lambda w$.
We define the projective line as set of equivalence classes

$$\mathbb{P}^1 := (k^2 \setminus \{0\})/\sim$$

and write $(x_0 \colon x_1)$ for the equivalence class of the vector (x_0, x_1). Then for any homogeneous polynomial $f \in k[T_0, T_1]$ the value $f(x_0 \colon x_1)$ is defined up to scalar multiple of $k^\times$. In particular whether f has a zero at $(x_0 \colon x_1)$ does not depend on the choice of representative (x_0, x_1). We define a topology on $\mathbb{P}^1$ by taking the sets

$$D(f) := \{(x_0 \colon x_1) \in \mathbb{P}^1 \mid f(x_0 \colon x_1) \neq 0\}$$

as a basis of topology, where f runs through all homogeneous polynomials in $k[x_0, x_1]$.
A homogeneous rational function of degree zero is by definition the quotient of two homogeneous polynomials of the same degree. If $\varphi = g/f$ is such a quotient, we consider it as a function $\varphi \colon D(f) \to k$ given by

$$\varphi(x_0 \colon x_1) = \frac{g(x_0, x_1)}{f(x_0, x_1)}.$$

Note that this is well-defined, i.e. the value $\varphi(x_0 \colon x_1)$ does not depend on the choice of the representative (x_0, x_1).
Given an open set $U \subset \mathbb{P}^1$, we call a function $\varphi \colon U \to k$ regular if for any $x \in U$ there exists an open neighbourhood $V \subset U$ of x such that $\varphi|_V$ is a homogeneous rational function of degree zero.
Denote by $\mathcal{O}_{\mathbb{P}^1}(U)$ the k-algebra of rational functions on U.

(a) Show that for $U_0 = \mathbb{P}^1 \setminus \{(0 \colon 1)\}$ and $U_1 = \mathbb{P}^1 \setminus \{(1 \colon 0)\}$ we have an isomorphism $(U_i, \mathcal{O}_{\mathbb{P}^1}|_{U_i}) \cong \mathbb{A}^1$. In particular, $(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1})$ is a prevariety (in fact, it is even a variety; but you do not need to show this).

(b) Prove that $\mathcal{O}_{\mathbb{P}^1}(\mathbb{P}^1) = k$ and conclude that any morphism of varieties $\mathbb{P}^1 \to \mathbb{A}^1$ is constant.

Exercise 10 (The Möbius transformation)**.** Prove that $\mathbb{P}^1$ is a GL_2-space with respect to the group action

$$\mathrm{GL}_2 \times \mathbb{P}^1 \to \mathbb{P}^1, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot (x_0 \colon x_1) = (ax_0 + bx_1 \colon cx_0 + dx_1)$$

and determine its orbits and the stabilizer of $(1 : 0)$.

Exercise 11 (Closure relations of orbits)**.** Let G be a linear algebraic group and X be a G-space. If $O, O' \subset X$ are two G-orbits in X we write $O \leq O'$ if O is contained in the closure of O'.

(a) Show that $\leq$ defines a partial order on the set of G-orbits in X.

(b) Determine the partial order for the GL_3-action on the variety N_3 of nilpotent 3×3-matrices.

Exercise 12 (The coordinate algebra of GL_n)**.** Give an explicit description of the comultiplication Δ, the antipode ι and the homomorphism ϵ corresponding to the identity element on the coordinate algebra of GL_n.

Deadline: Friday, 10th November, 2017

If you have any questions regarding the exercises, please send an email to hamacher@ma.tum.de. The exercise classes are Fridays, 10-12 in room MI 02.08.020. Further information about our lectures and exercises are available under `http://www-m11.ma.tum.de/viehmann/viehmann-linear-algebraic-groups/`.