Technische Universität München
Zentrum Mathematik

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Winter term 2017/18
Exercise sheet 4

Linear algebraic groups (MA 5113)

Exercise 13 (Jordan decomposition in the product)**.** Let G and G' be two linear algebraic groups. Show that for any element $(g, g') \in G \times G'$ the Jordan decomposition is given by

$$(g, g')_s = (g_s, g'_s)$$
$$(g, g')_u = (g_u, g'_u).$$

Project: Finite linear algebraic groups. Let k be an algebraically closed field. Recall that any finite set $G \subset k^n$ is Zariski closed and thus an affine variety. Since the topology is discrete, any subset $U \subset G$ is open and closed, thus one gets

$$\mathcal{O}_G(U) = k[X_1, \ldots, X_n]/\mathcal{I}_{k^n}(U) \cong k^U.$$

In particular, any finite set G can be considered as an affine variety by equipping it with the discrete topology and the structure sheaf given by $\mathcal{O}_G(U) = k^U$.

Exercise 14 (Finite groups as linear algebraic groups)**.** Let G be a finite (abstract) group. Show that, when equipped with the structure of an affine variety as above, G becomes a linear algebraic group. Determine the comultiplication Δ, the antipode ι and the morphism $\epsilon\colon k[G] \to k[G]$ given by the identity element.

Exercise 15 (Jordan decomposition in finite groups)**.** Given a finite linear algebraic group G as above determine the sets of semi-simple/unipotent elements G_s and G_u and describe the Jordan decomposition of a given element $g \in G$.

Exercise 16 (Quotients by finite groups)**.** Let X be an affine G-space, where G is a finite linear algebraic group. We denote by $A := k[X]$ the coordinate algebra and by $A^G \subset A$ the subalgebra of invariants under the G-action given by $g.f(x) := f(g^{-1}x)$ for any $f \in A$.

(a) Show that A is finite over A^G.
(b) Conclude that A^G is an affine k-algebra.
(c) Let $X/G = \operatorname{Spec}_{\max} A^G$ the affine variety associated to A^G. Prove that the underlying topological space is the quotient space of X by the G-action, i.e. the morphism $p\colon X \to X/G$ corresponding to $A^G \hookrightarrow A$ identifies X with the set of G-orbits on X and a set $U \subset X$ is open if and only if $p^{-1}(U)$ is open.

Deadline: Friday, 17th November, 2017

If you have any questions regarding the exercises, please send an email to hamacher@ma.tum.de. The exercise classes are Fridays, 10-12 in room MI 02.08.020. Further information about our lectures and exercises are available under `http://www-m11.ma.tum.de/viehmann/viehmann-linear-algebraic-groups/`.