Technische Universität München
Zentrum Mathematik

Winter term 2017/18
Exercise sheet 5

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Linear algebraic groups (MA 5113)

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Announcement:  The Wednesday lecture has been rescheduled.
It will take place 12.00(s.t.!) - 13.30 from now on.
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Exercise 17 (Tangent space of the product variety)**.** Let k be an algebraically closed field.

(a) Let X be an affine variety over k and $X_1 \subset X$ a closed subvariety with $\mathcal{I}_X(X_1) = (f_1, \dots, f_m)$. Show that for every $x \in X_1$,

$$T_x X_1 = \{\partial \in T_x X \mid \forall i : \partial f_i = 0\}$$

(b) Let X, Y be two varieties over k, $x_0 \in X, y_0 \in Y$ two points. Denote

$$\iota_X : X \hookrightarrow X \times Y, x \mapsto (x, y_0)$$
$$\iota_Y : Y \hookrightarrow X \times Y, y \mapsto (x_0, y)$$

Prove that

$$d\iota_X \oplus d\iota_Y : T_x X \oplus T_y Y \to T_{(x,y)}(X \times Y)$$

is an isomorphism.

Exercise 18 (Differentials of certain morphisms)**.** Let G be a linear algebraic group over k with multiplication $m\colon G \times G \to G$ and inverse $i\colon G \to G$. Show that, using the above identification $T_{(e,e)}(G \times G) = T_e G \oplus T_e G$ from the previous exercise, we have

$$dm(X, Y) = X + Y,$$
$$di(X) = -X.$$

Exercise 19 (Examples of Lie algebras)**.** Assume that $\operatorname{char} k \neq 2$. Determine the Lie algebras of the following linear algebraic groups over k and determine their dimension.

(a) $T := \{(a_{i,j}) \in \mathrm{GL}_n \mid \forall i \neq j : a_{i,j} = 0\}$

(b) $\mathrm{O}_n := \{A \in \mathrm{GL}_n \mid A \cdot A^T = 1\}$

(c) $\mathrm{Sp}_{2n} := \{A \in \mathrm{GL}_{2n} \mid A \cdot J \cdot A^T = J\}$, where $J = ((-1)^{\delta_{j>n}} \cdot \delta_{i,2n+1-j})_{i,j}$

You may use without proof that the ideal given by the equations in (b) and (c) is a radical ideal.

Representations of Lie algebras: For any finite dimesensional k-vector space V, we denote by $\mathfrak{gl}(V)$ the Lie algebra of a $\mathrm{GL}(V)$ considered as a linear algebraic group. A morphism of Lie algebras $\mathfrak{g} \to \mathfrak{gl}(V)$ is called a *representation* of $\mathfrak{g}$. One way to obtain a representation of the Lie algebra $\mathfrak{g}$ of a linear algebraic group G is to consider the differential $d\phi\colon \mathfrak{g} \to \mathfrak{gl}(V)$ of a rational representation $\phi\colon G \to \mathrm{GL}(V)$. Note that in particular $\mathfrak{g}$ "acts" on V, since $\mathfrak{gl}(V) \cong \mathrm{End}(V)$. One also has the following alternative description of this action. If we consider the representation ϕ as a morphism of varieties

$$f\colon G \times V \to V, (g, v) \mapsto \phi(g) \cdot v$$

and take the differential at $(e, 0)$ we obtain

$$(df)_{(e,0)}\colon \mathfrak{g} \times V \to V.$$

One can show that this "action" is identical to the one above.

Exercise 20 (Tensorial constructions)**.** Let G, G_1, G_2 be linear algebraic groups with respective Lie algebras $\mathfrak{g}, \mathfrak{g}_1, \mathfrak{g}_2$. Moreover assume that we are given rational representations $\phi\colon G \to \mathrm{GL}(V)$ and $\phi_i\colon G_i \to \mathrm{GL}(V_i)$ $(i = 1, 2)$.

(a) Consider the (rational) representations

$$\phi_1 \oplus \phi_2\colon G_1 \times G_2 \to \mathrm{GL}(V_1 \oplus V_2), (\phi_1 \oplus \phi_2)(g_1, g_2)(v_1, v_2) = (\phi_1(g_1)(v_1), \phi_2(g_2)(v_2))$$
$$\phi_1 \otimes \phi_2\colon G_1 \times G_2 \to \mathrm{GL}(V_1 \otimes V_2), (\phi_1 \otimes \phi_2)(g_1, g_2)(v_1 \otimes v_2) = \phi_1(g_1)(v_1) \otimes \phi_2(g_2)(v_2)$$

Show that the corresponding representations of the Lie algebra $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ are given by

$$d(\phi_1 \oplus \phi_2)(X_1, X_2)(v_1, v_2) = (d\phi_1(X_1)(v_1), d\phi_2(X_2)(v_2))$$
$$d(\phi_1 \otimes \phi_2)(X_1, X_2)(v_1 \otimes v_2) = d\phi_1(X_1)(v_1) \otimes v_2 + v_1 \otimes d\phi_2(X_2)(v_2).$$

(b) Similarly, consider the rational representations

$$\oplus^n \phi\colon G \to \mathrm{GL}(V^n), (\oplus^n \phi)(g)(v_1, \ldots, v_n) = (\phi(g)(v_1), \ldots, \phi(g)(v_n))$$
$$\otimes^n \phi\colon G \to \mathrm{GL}(V^{\otimes n}), (\otimes^n \phi)(g)(v_1 \otimes \ldots \otimes v_n) = \phi(g)(v_1) \otimes \ldots \otimes \phi(g)(v_n).$$

Conclude that their differentials are given by

$$d(\oplus^n \phi)(X)(v_1, \ldots, v_n) = (d\phi(X)(v_1), \ldots, d\phi_n(X)(v_n)).$$
$$d(\otimes^n \phi)(X)(v_1 \otimes \ldots \otimes v_n) = \sum_{i=1}^{n} v_1 \otimes \ldots \otimes d\phi(X)(v_i) \otimes \ldots \otimes v_n.$$

(c) The *n-th exterior algebra* $\bigwedge^n V$ of a k-vector space V is defined as

$$V^{\otimes n} / \operatorname{span}(S)$$

where $S \subset V^{\otimes n}$ is the set of tensors $v_1 \otimes \ldots \otimes v_n$ such that there exist i, j with $v_i = v_j$. The image of $v_1 \otimes \ldots \otimes v_n$ is denoted by $v_1 \wedge \ldots \wedge v_n$. This vector space is also called *alternating product* because of the rule

$$v_1 \wedge \ldots \wedge v_n = (-1)^{\operatorname{sign} \sigma} \cdot v_{\sigma(1)} \wedge \ldots \wedge v_{\sigma(n)}$$

for any permutation $\sigma \in \mathfrak{S}_n$.
Now consider the rational representation

$$\wedge^n \phi\colon G \to \mathrm{GL}(\bigwedge^n V), (\wedge^n \phi)(g)(v_1 \wedge \ldots \wedge v_n) = \phi(g)(v_1) \wedge \ldots \wedge \phi(g)(v_n).$$

Prove that

$$d(\wedge^n \phi)(X)(v_1 \wedge \ldots \wedge v_n) = \sum_{i=1}^{n} v_1 \wedge \ldots \wedge d\phi(X)(v_i) \wedge \ldots \wedge v_n.$$

Deadline: Friday, 24th November, 2017

If you have any questions regarding the exercises, please send an email to hamacher@ma.tum.de. The exercise classes are Fridays, 10-12 in room MI 02.08.020. Further information about our lectures and exercises are available under `http://www-m11.ma.tum.de/viehmann/viehmann-linear-algebraic-groups/`.