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Zentrum Mathematik – M11

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An introduction to perfectoid spaces

The theory of perfectoid spaces was invented by Peter Scholze to compare geometric objects over local fields of mixed characteristic with geometric objects over local fields of equal characteristic. It was recently discovered that many natural objects can be interpreted as perfectoid spaces, e.g. Shimura varieties with infinite level structure at p, which we want to study in the upcoming semester. For a more detailed motivation and overview of the theory of perfectoid spaces see [6].

The aim of this (mini-)course is to understand the definition of perfectoid spaces and the construction of the tilting functor, which relates perfectoid spaces over local fields of mixed characteristic with perfectoid spaces over local fields of equal characteristic. As it would take about 1-2 semesters to cover everything in full detail, we will abstain from giving most proofs and leave out technical details.

The participants are expected to have a strong algebraic background and a good understanding of local fields. If you are not familiar with rigid geometry, we advise you to read the overview article of Schneider [4].

Each talk should take 90 minutes (thus a $1/2$ talk should take 45 minutes).

Day 1

1. Almost mathematics (1 talk)
([5], section 4) We fix a non-archimedean field K and denote by $(K^\circ, \mathfrak{m})$ its valuation ring. The aim of this talk is to introduce the concept of almost mathematics over K°, which considers roughly speaking objects over K° only up to $\mathfrak{m}$-torsion.
Note: Explain (and prove) 4.1-4.6 in great detail following [1] § 2.2. Note that [1] covers the subject in greater generality than we need; using their notation we have $\mathfrak{m} = \tilde{\mathfrak{m}}$ and condition (A) of 2.1.6 holds. You may skip the proofs of 4.7-4.17, but it would be nice to see some non-trivial proof in almost ring theory. Also note that an explicit description of almost finitely generated modules can be found in [7] ch. 3.

2. Perfectoid fields ($1/2$ talk)
([5], section 3) We introduce perfectoid fields, which will be the base fields of perfectoid spaces. Also, we discuss a basic example of the tilting process.
Note: Illustrate the content by giving examples for perfectoid fields (and their tilts) and also give the example of the tilting process in Theorem 3.7 (ii) explained in [6] page 3.

3. Perfectoid algebras (1/2 talk)
([5], section 5 until 5.21) The aim of this talk is to introduce perfectoid algebras and sketch the construction of the algebra version of the tilting functor. Please follow this outline:

- Define perfectoid K-algebras. Give the standard example $K\langle X_1^{1/p^\infty}, \ldots, X_n^{1/p^\infty}\rangle$.
- Prove Prop. 5.8 and Prop. 5.9.
- Define perfectoid $K^{\circ a}$-algebras and perfectoid $K^{\circ a}/\varpi$-algebras.
- State Theorem 5.2 and explain the different constructions (without proof). Also explain the tilting process for the standard example, i.e. prove Prop. 5.20.
- Also state Lemma 5.21
- If you have any time left, you can explain some steps of the proof in greater detail or give the proof of Lemma 5.21.

4. Finite étale extensions (1/2 talk)
([5], 5.22-5.25) Using the tilting procedure constructed in the previous talk, we prove a weaker version of the equivalence of finite étale covers under the additional assumption on the almost level. We will prove that this assumption always holds in talk 11. The result is strong enough to finally finish the proof of Theorem 3.7.

Day 2

5. Valuations (1/2 talk)
([8], §1-2; [2], 1.1.1-1.1.8) In this talk we recall valuations and valuation rings and introduce the valuation spectrum.
Note: You should be brief about valuations and valuation rings, but make sure to include the following content in your talk:

- The definition of convex subgroups and the rank of a valuation.
- The correspondence of valuations of a field k up to equivalence and valuation rings with quotient field k.
- The correspondence between extensions of valuation rings, prime ideals and convex subgroups of the value group (see [8], Prop. 2.16.

Also, give some examples, e.g. $\operatorname{Spv}\mathbb{Q}$ and $\operatorname{Spv}\mathbb{Z}$.

6. Adic spaces (4/3 talk)
([5] 2.6 - 2.19, 2.25 - 2.29) We introduce adic spaces over a non-archimidean field. Adic spaces are a generalisation of rigid-analytic varieties, which allows us to consider spaces associated to Tate k-algebras which are not topologically of finite type. This generalisation is necessary as most perfectoid algebras are "too big" to be topologically of finite type.
Note: It might be instructional to see that R° is always integrally closed and that $R^{\circ\circ}$ is a radical ideal (see e.g. [8] Prop. 5.30). The time will not suffice to give all proofs, but make sure that you at least sketch the proof of 2.10 and give the full proofs of 2.25-2.29.

7. Rigid-analytic varieties as adic spaces - examples (1/2 talk)
([5], 2.20-2.21) We (very) briefly recall rigid-analytic varieties and their admissible topology. There exists a canonical fully faithful functor from the category of rigid analytic varieties into the category of adic space. We give a detailed example of the unit ball $\mathbb{B}^1$ as an adic space and also construct $\mathbb{A}^n$ and $\mathbb{P}^n$ as adic spaces.

8. Étale morphisms of adic spaces (2/3 talk)
([7], ch. 4 until the corollary on p.18) Here we introduce étale morphisms of adic spaces and discuss some basic properties.
Note: A more detailed exposition as in the lecture notes can be found in Huber's book [3], ch. 1 and ch. 2.2.

Day 3

9. Perfectoid spaces: Tilting (2/3 talk)
([5], section 6.1 - 6.8, 6.15 - 6.18) In this and the next talk we introduce perfectoid spaces as the full subcategory of adic spaces which have an affinoid cover of adic spaces associated to perfectoid algebras. These spaces have nice geometric properties and it is possible to globalize the tilting functor of talk 3 to an equivalence between categories of perfectoid spaces. In this talk we define the tilt of perfectoid spaces and show that it is a homeomorphism as well as that it defines tilts on the structure (pre-)sheaves.
Note: Skip the (very technical) proofs of Lemma 6.4 and 6.5.

10. Perfectoid spaces: The structure sheaf (1/2 talk)
([5], 6.10 - 6.14) We prove that the structure presheaves of an affinoid perfectoid space are actually sheaves and that the cohomology groups of $\mathcal{O}_X^+$ are almost zero in degrees greater than one.

11. Étale topology of perfectoid spaces (1 talk)
([5], section 7 until 7.13) The main theorem of this talk is that the tilting procedure of perfectoid spaces also induces an equivalence of their étale sites. This is the strong and globalized form of the equivalence proven in talk 4.

12. Limits of étale topoi (1/3 talk)
([5], rest of section 7) One has the notion that perfectoid spaces are "large" objects, while one often considers smaller objects e.g. spaces which are topologically of finite type. By writing a perfectoid space as a limit of smaller object, one gets a comparison of their étale topoi. If one is lucky, one might even define the tilting procedure on smaller objects as well.
Note: Give the projective space as example (cf. [5], Prop. 8.5)

References

[1] O. Gabber, L. Ramero: Almost ring theory, Lecture Notes in Mathematics, **1800**

[2] R. Huber: Bewertungsspektrum und rigide Geometrie

[3] R. Huber: Étale cohomology of rigid analytic varieties and adic sapces.

[4] P. Schneider: Basic notions of rigid algebraic geometry, `www.math.uni-muenster.de/u/pschnei/publ/pap/rigid.ps`

[5] P. Scholze: Perfectoid spaces, `http://www.math.uni-bonn.de/people/scholze/PerfectoidSpaces.pdf`

[6] P. Scholze: Perfectoid spaces, a survey, http://www.math.uni-bonn.de/people/scholze/CDM.pdf

[7] Informal lecture notes of the course "p-adic Hodge theory" given by Peter Scholze in the summer term 2012.

[8] T. Wedhorn: Adic spaces, http://www2.math.uni-paderborn.de/fileadmin/Mathematik/People/wedhorn/Lehre/AdicSpaces.pdf