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Zentrum Mathematik – M11

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Shimura varieties of infinite level at p

In this seminar we follow the third section Scholze's paper [Sch]. Perfectoid spaces studied in the pre-seminar before allow us to study Shimura varieties of all levels at p simultaneously. Let A_{g,K_pK^p} be the moduli space of principally polarised g-dimensional abelian varieties (over $\operatorname{Spec}\mathbb{Z}$), with level structure K_pK^p where $K_p \subseteq \mathrm{Sp}_{2g}(\mathbb{Q}_p)$ stands for the p-part of the level and $K^p \subseteq \mathrm{Sp}_{2g}(\mathbb{A}_f^p)$ for the prime-to-p part of the level. Let A^*_{g,K_pK^p} denote its minimal Baily-Satake compactification. In particular, for $g = 1$, the space A_{g,K_pK^p} is the classical modular curve, and A^*_{g,K_pK^p} its compactification by adjoining cusps.

We will see that if K^p is small enough, there is a perfectoid space $A^*_{g,K^p}/\mathbb{C}_p$, which is in a sense the limit over all level structures K_p of the spaces A^*_{g,K_pK^p}:

$$A^*_{g,K^p} = \varprojlim_{K_p \subseteq \mathrm{Sp}_{2g}(\mathbb{Q}_p)} (A^*_{g,K_pK^p})^{\mathrm{ad}}$$

(here ad denotes the adic completion). Then we will construct and study properties of the *Hodge-Tate period map*

$$\pi_{HT} \colon A^*_{g,K_p} \to \mathscr{F}\ell,$$

where $\mathscr{F}\ell$ is the flag variety parametrizing totally isotropic subspaces of $\mathbb{C}_p^{2g}$. This map is the p-adic analogue of the following well-known complex situation: for a congruence subgroup $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$, let $\mathscr{M}^*_\Gamma$ denote the (compactified) modular curve associated to it. Then it is universally covered by he upper half plane $\mathbb{H} \twoheadrightarrow \mathscr{M}_\Gamma$ and we can see $\mathbb{H}$ as the limit over the $\mathscr{M}_\Gamma$'s for Γ varying over all arithmetic subgroups of $\mathrm{SL}_2(\mathbb{Z})$. Then the embedding $\mathbb{H} \hookrightarrow \mathbb{P}^1(\mathbb{C})$ should be seen as the classical analogue to the map π_{HT} defined above. Remark that in contrast to the complex case π_{HT} is not injective.

Motivation: One reason for studying this morphism, is the possibility to relate the étale cohomology of A^*_{g,K^p}, which is equal to the limit of the étale cohomology of A^*_{g,K_pK^p} for varying K_p, to the (topological) cohomology of A^*_{g,K^p}, which then can be expressed in terms of cuspidal forms modulo p.

The motivation for all this is the following conjecture which is a 'mod p' version of the global Langlands correspondence. To explain it, let F be a number field, $K \subseteq \mathrm{GL}_n(\mathbb{A}_f)$ a (sufficiently small compact open subgroup), $K_\infty \subseteq \mathrm{GL}_n(K \times \mathbb{R})$ a maximal compact subgroup, and X_K a locally symmetric variety defined by:

$$X_K := \mathrm{GL}_n(F)\backslash(\mathrm{GL}_n(F \times \mathbb{R})/K_\infty\mathbb{R}_{>0} \times \mathrm{GL}_n(\mathbb{A}_{F,f})/K).$$

For example in the special case $F = \mathbb{Q}$, $n = 2$, X_K are the complex points of a modular curve. In general X_K is only a real manifold without a complex structure. The singular cohomology $\mathrm{H^*(X_K, \mathbb{C})}$ of these varieties is acted on by Hecke operators and Franke showed in [Fr] that one can describe this spaces (with the Hecke action) in terms of automorphic representations of $\mathrm{GL}_n(\mathbb{A}_F)$. By the Langlands philosophy one expects that there are representations of G_F associated with systems of Hecke operators occurring in $\mathrm{H}^*(\mathrm{X}_\mathrm{K}, \mathbb{C}) \cong \mathrm{H}^*(\mathrm{X}_\mathrm{K}, \overline{\mathbb{Q}_\mathrm{p}})$, after choosing an isomorphism $\iota\colon \mathbb{C} \cong \overline{\mathbb{Q}_p}$.

If one look at the cohomology groups with coefficients in $\overline{\mathbb{Z}_p}$ or $\overline{\mathbb{F}_p}$, it turns out that the first one can have a lot of p-torsion, such that $\mathrm{H}^*(\mathrm{X}_\mathrm{K}, \overline{\mathbb{F}_\mathrm{p}})$ has a much bigger dimension than $\mathrm{H}^*(\mathrm{X}_\mathrm{K}, \overline{\mathbb{Q}_\mathrm{p}})$. Nevertheless, one conjectures still that there are associated Galois representations.

Conjecture 0.1. *For any system of Hecke eigenvalues on* $\mathrm{H}^*(\mathrm{X}_\mathrm{K}, \overline{\mathbb{F}_\mathrm{p}})$ *there is a continuous semisimple n-dimensional representation of* G_F *over* $\overline{\mathbb{F}_p}$*, such that the eigenvalues of the Hecke operators agree with the traces of Frobenius elements.*

This conjecture was made by many people, in particular Grunewald, Ash, Figueiredo. The final goal and the motivation for this seminar is the following theorem:

Theorem 0.2. *The conjecture 0.1 is true if F is a totally real or a CM field.*

The rough idea of Scholze's approach to this theorem (which however will not be part of this seminar) is to realize the cohomology of X_K in the boundary of a $GU(n,n)$-Shimura variety. Then, using p-adic deformation, one obtains a cusp form on $GU(n,n)$, where associated Galois representations are known to exist. Perfectoid spaces enter the proof, when one starts to study the cohomology of the involved Shimura varieties, now having the advantage of doing this at an infinite level at p.

References

[Fr] Franke J.: *Harmonic analysis in weighted L^2-spaces*, Ann. Sci. Ecole Norm. Sup. (4), 31(2):181-279, 1998.

[Sch] Scholze P.: *On torsion in cohomology of locally symmetric varieties*, `http://www.math.uni-bonn.de/people/scholze/Torsion.pdf`