Technische Universität München
Zentrum Mathematik

Summer term 2017
Exercise sheet 1

Prof. Dr. Eva Viehmann
Dr. Paul Hamacher

Linear algebraic groups (MA 5113)

Exercise 1 (Linear algebraic groups as varieties)**.** Let k be an algebraically closed field with $\operatorname{char} k \neq 2$. Determine whether the following subsets of $\mathbb{A}_k^{n\times n}$ are closed or open and which are irreducible.

(a) $\mathrm{GL}_n := \{A \in k^{n\times n} \mid A \text{ is invertible}\}$
(b) $\mathrm{SL}_n := \{A \in k^{n\times n} \mid \det(A) = 1\}$
(c) $\mathrm{O}_n := \{A \in k^{n\times n} \mid A \cdot A^t = I_n\}$

Exercise 2 ((Non-)Examples of ringed spaces)**.** Let X be a topological space. Which of the following functions $\mathcal{O}$ (together with the obvious restriction maps) are sheaves on X for every topological space X ?

(a) $\mathcal{O}(U) = \mathbb{C}^U$ (i.e. the $\mathbb{C}$-algebra of maps $f\colon U \to \mathbb{C}$)
(b) $\mathcal{O}(U) = \{f \in \mathbb{C}^U \mid f \text{ is continuous}\}$
(c) $\mathcal{O}(U) = \{f \in \mathbb{C}^U \mid f \text{ is constant}\}$
(d) $\mathcal{O}(U) = \{f \in \mathbb{C}^U \mid f \text{ is bounded}\}$

Exercise 3 (Isomorphisms of sheaves)**.** Let X be a topological space, $\mathfrak{B}$ be a basis of its topology and $\varphi\colon \mathcal{O} \to \mathcal{O}'$ be a morphism of sheaves on X. Show that the following are equivalent.

(a) φ is an isomorphism.
(b) For every $U \in \mathfrak{B}$ the induced morphism $\mathcal{O}(U) \to \mathcal{O}'(U)$ is an isomorphism.
(c) For every $x \in X$ the induced morphism of stalks $\mathcal{O}_x \to \mathcal{O}'_x$ is an isomorphism.

Exercise 4 (Examples of tensor products)**.** Let R be a ring and M be an R-module. Prove that

(a) $R/I \otimes_R R/J \cong R/(I+J)$ for any ring R and ideals $I, J \trianglelefteq R$,
(b) $M \otimes_R R/I \cong M/I \cdot M$ for any ideal $I \trianglelefteq R$ and
(c) $M \otimes_R R[S^{-1}] \cong M[S^{-1}]$ for any multiplicative subset $S \subset R$.

Deadline: Friday, 27 October, 2017

If you have any questions regarding the exercises, please send an email to hamacher@ma.tum.de. The first exercise class will be on Friday, 20th October at 14.15 in room 03.10.011; time and place of the other exercise classes will be discussed after the second lecture. Further information about our lectures and exercises are available under `http://www-m11.ma.tum.de/viehmann/viehmann-linear-algebraic-groups/`.