Technische Universität München
Zentrum Mathematik

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Winter term 2020/21
Exercise sheet 1

Abelian varieties (MA 5115)

Exercise 1 (Algebraic groups as functors)**.** Let S be a scheme and X be a scheme over S. We denote by

$$\begin{aligned} h^X &:= (Sch/k)^o \to (Sets) \\ T &\mapsto h_X(T) := \mathrm{Hom}_{(Sch/S)}(T, X) \\ (f\colon T \to T') &\mapsto (h_X(T') \to h_X(T), g \mapsto g \circ f) \end{aligned}$$

its functor of points. Recall that by the Yoneda lemma $X \mapsto h^X$ is functorial in X and that for any two S-schemes X, Y the natural map $\mathrm{Hom}_{(Sch/S)}(X, Y) \mapsto \mathrm{Hom}(h^X, h^Y)$ is a bijection.
We had seen in the that if G is an algebraic group then $G(k) = h^G(\operatorname{Spec} k)$ has a natural group structure (in fact, we defined it this way). We will show that a more general version of this fact can even be used to give an alternative and more managable definition of a group scheme.

(a) Calculate $X(R) := h_X(R)$ where R is a ring and X any of the schemes

$$\begin{aligned} \mathrm{SL}_n &:= \operatorname{Spec} \mathbb{Z}[X_{1,1}, \ldots, X_{n,n}]/(\det -1) \\ \mathrm{GL}_n &:= \operatorname{Spec} \mathbb{Z}[X_{1,1}, \ldots, X_{n,n}]_{\det}. \end{aligned}$$

(b) Let G be a group scheme over S and m, i, e as in Definition 1.1. Show that for any S-scheme T the morphisms m, i, e define a group structure on $h^G(T)$. Show moreover that this group structure is functorial in T, i.e that h^G together with the group structure defined above defines a functor $(Sch/S)^o \to (Groups)$.

(c) Show that the converse is also true. Let G be an S-scheme such that the functor of points h^G can be lifted to a functor $h^G_{grp}\colon (Sch/S)^o \to (Groups)$. Show that there exist (unique) morphisms m, i, e making G into a group scheme such that the functor constructed in (b) coincides with h^G_{grp}.

Exercise 2 (Integral morphisms)**.** In this exercise we will prove the last missing piece of the claim "proper+affine = finite" for morphisms of schemes.
Let $\varphi^*\colon R \to A$ be an finite ring homomorphism. Prove that the corresponding morphism of schemes $\varphi\colon \operatorname{Spec} A \to \operatorname{Spec} R$ is proper, using the following outline of the proof.

(a) Show that φ is separated and of finite type.

(b) In order to prove that φ is closed, recall (from Algebra II) that φ satisfies going up. Then conclude that it is closed.

(c) Show that since every finite ring homomorphism induces a closed morphism of schemes, it must even be universally closed.

Deadline: Monday, 9th November, 2020

If you have any questions regarding the exercises, please send an email to hamacher@ma.tum.de. The first exercise class will be on Wednesday, 11th November at 13.15 in the same Zoom room as the lecture.