Technische Universität München
Zentrum Mathematik

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Winter term 2020/21
Exercise sheet 10

Abelian varieties (MA 5115)

As usual, we work over an algebraically closed field k.
This week we will work on constructing the dual Abelian variety. In exercise 1 we construct the dual, using the (difficult!) statement that the Picard functor is representable. If one wants to construct the dual Abelian variety without using the representability of the Picard functor, one needs a general geometric statement, which we prove in Exercise 2.

Exercise 1. Let A be an Abelian variety.

(a) Prove that $\mathrm{Pic}_{A/k}$ is a group scheme.

(b) Prove that $\mathrm{Pic}^0_{A/k}(k) \subset \mathrm{Pic}_{A/k}(k)$ is a closed subgroup and in particular defines a closed reduced subgroupscheme $\mathrm{Pic}^0_{A/k}$

(c) Prove that $\mathrm{Pic}^0_{A/k}$ is an Abelian variey

Exercise 2. Let X, Y be irreducible varieties and denote their function fields by $K \coloneqq k(Y), L \coloneqq k(X)$. Any dominant morphism $\varphi\colon X \to Y$ then induces an embedding $\varphi^*\colon K \hookrightarrow L$.

(a) φ is called separable, if L/K is separable.

(b) φ is called birational, if $K = L$. Equivalently, there exist $U \subset X, V \subset V$ open such that $\varphi\colon U \to V$ is an isomorphism. In fact any isomorphism $k(X) \cong k(Y)$ induces such a morphism between open (dense) subsets.

The aim of this exercise is to show that any finite morphism $\varphi\colon X \to Y$ such that $X(k) \to Y(k)$ is bijective is birational. You may follow the following outline in your proof.

(a) First show that if $U \subset X, V \subset Y$ are open we may replace X by U and Y by $\varphi(U)$ or X by $\varphi^{-1}(V)$ and Y by V without changing the claim to be proven.

(b) Using (a) make the following reduction steps
   (a) $Y = \operatorname{Spec} R$, $X = \operatorname{Spec} A$ are affine. In particular $K = \mathrm{Frac}(R), L = \mathrm{Frac}(A)$
   (b) We have $L \cong K[x]/(f)$ with $f \in R[x]$.
   (c) We have $A = R[x]/(f)$.

(d) Now deduce that by the bijectivity, f must be a linear polynomial and thus $K \cong L$.

Deadline: Monday, 1st February, 2021