Technische Universität München
Zentrum Mathematik

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Winter term 2020/21
Exercise sheet 11

Abelian varieties (MA 5115)

As usual, we work over an algebraically closed field k.
This week we will two useful statement when working with Abelian varieties.

Exercise 1. Let $\varphi\colon A \to B$ be an isogeny of Abelian varieties of degree n. We assume $\operatorname{char} k = 0$, in particular φ is étale (see Exercise sheet 9 for explicit constructions of étale isogenies).

(a) Prove that there exists a unique isogeny $\varphi'\colon B \to A$ such that $\varphi' \circ \varphi = [n]_A$.

(b) Show that with φ' satisfying (a), we also have $\varphi \circ \varphi' = [n]_A$.

(c*) Can you show above exercises when $\operatorname{char} k > 0$? You may assume wihtout prove that multiplication with n equals zero on $(\ker \varphi)_{sch}$.

Exercise 2. Let A be an Abelian variety.

(a) Let T be a connected variety and $\mathscr{L} \in \operatorname{Pic}(A \times T)$ such that there exists $t \in T(k)$ such that

$$\mathscr{L}_t := \mathscr{L}|_{A\times\{t\}} \in \operatorname{Pic}^0(A) \tag{0.1}$$

Show that (0.1) must be true for all $t \in T(k)$.

(b) Conclude that $\operatorname{Pic}^0_{A/k}$ is the unit component of $\operatorname{Pic}_{A/k}$.

Remark: Part (b) shows that our notation is consistent with earlier notation: For an algebraic group G we denoted by G^0 its unit component. It is also true that the moduli space $\operatorname{Pic}^0_{C/k}$ of degree zero line bundles on a smooth projective curve C is the unit component of the Picard variety $\operatorname{Pic}_{C/k}$. Thus being in the unit component of the Picard variety can be thought of being the correct generalisation of "degree zero line bundle" to smooth projective varieties of dimension greater than one.

Deadline: Monday, 8th February, 2021