Technische Universität München
Zentrum Mathematik

Dr. Paul Hamacher

Winter term 2020/21
Exercise sheet 3

Abelian varieties (MA 5115)

Exercise 1. We have seen in the lecture that every elliptic curve can be written in the form $\mathbb{C}/\Lambda$ where Λ is a full lattice of $\mathbb{C}$. We will use this description to parametrise the isomorphism classes of elliptic curves.

(a) Prove that $\mathbb{C}/\Lambda$ and $\mathbb{C}/\Lambda'$ are isomorphic if and only if Λ and Λ' are homothetic to each other, i.e. $\Lambda' = a\Lambda$ for an $a \in \mathbb{C}$.

(b) Prove that every full lattice $\Lambda \in \mathbb{C}$ is homothetic the lattice $\langle 1, \tau \rangle_{\mathbb{Z}}$ spanned by 1 and τ for some $\tau \in \mathbb{C}$ with $\operatorname{Im} \tau > 0$, i.e. positive imaginary part.

(c) Prove that two lattices $\langle 1, \tau \rangle_{\mathbb{Z}}$ and $\langle 1, \tau' \rangle_{\mathbb{Z}}$ are homothetic if and only if there exists $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ such that $\tau' = \frac{a\tau+b}{c\tau+d}$.

Altogether, we have shown that the quotient $\mathbb{H}/\,\mathrm{SL}_2(\mathbb{Z})$ parametrises the moduli space of elliptic curves over $\mathbb{C}$, where $\mathbb{H} := \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$ denote the upper half plane.

Exercise 2. Let A_1, A_2 be two Abelian manifolds over $\mathbb{C}$ of dimensions g_1, g_2.

(a) Show that $\operatorname{Hom}(A_1, A_2)$ is a free $\mathbb{Z}$-module of rank no larger than $4g_1g_2$. Can you find a better upper bound ?

(b) Now assume that A_1, A_2 are elliptic curves, i.e. $g_1 = g_2 = 1$. Find an example where $\operatorname{Hom}(A_1, A_2) = 0$, $\operatorname{Hom}(A_1, A_2) \cong \mathbb{Z}$ and an example where $\operatorname{Hom}(A_1, A_2) \cong \mathbb{Z}^2$.

Deadline: Monday, 23rd November, 2020