Technische Universität München
Zentrum Mathematik

Winter term 2020/21
Exercise sheet 4

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Abelian varieties (MA 5115)

Exercise 1. An important special case of a homomorphism of Abelian varieties is an isogeny, that is a surjective homomorphism $\varphi\colon A \to B$ with finite kernel. In this exercise, we will study isogenies over the base field $\mathbb{C}$.

(a) Let A be a complex torus, and $G \subset A$ be a finite group. Show that up to there exists an isogeny $\varphi\colon A \twoheadrightarrow A'$ with $\ker\varphi = G$ and that it satisfies the following universal property. For every homomorphism of complex tori $\psi\colon A \to B$ with $G \subset \ker\psi$, there exists a unique homomorphism $\bar{\psi}\colon A' \to B$ such that the following diagram commutes.

$$\begin{tikzcd} A \arrow[r, "\psi"] \arrow[d, "\varphi"] & B \\ C. \arrow[ur, dashed, "\bar{\psi}"'] & \end{tikzcd}$$

This holds true even when B is any complex Lie group (you don't have to prove this), thus φ defines the quotient object $A \to A/G$.

(b) Let $\varphi\colon A \to A'$ be an isogeny and let $n := \ker\varphi$. Show that there exists a unique isogeny $\varphi^\vee\colon A' \to A$ such that $\varphi^\vee \circ \varphi\colon A \to A$ is the multiplication by n. Also show that $\varphi^{\vee\vee} = \varphi$.

Exercise 2. Prove the following handy facts about line bundles. In the following let X be a variety (or scheme).

(a) Let $\mathscr{L}_1, \mathscr{L}_2 \in \operatorname{Pic}(X)$ and $\alpha\colon \mathscr{L}_1 \to \mathscr{L}_2$ be a surjective morphism of $\mathscr{O}_X$-modules. Show that α is an isomorphism.

(b) Show that a line bundle $\mathscr{L} \in \operatorname{Pic}(X)$ is trivial if and and only if it has a section $s \in \mathscr{L}(X)$ such that $V(s) = \emptyset$.

(c) Now assume that X is a proper variety. Prove that a line bundle $\mathscr{L} \in \operatorname{Pic}(X)$ is trivial if and and only if both $\mathscr{L}(X), \mathscr{L}^{-1}(X) \neq \{0\}$ (Hint: $O_X(X) = k$)

Deadline: Monday, 30th November, 2020