Technische Universität München
Zentrum Mathematik

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Winter term 2020/21
Exercise sheet 7

Abelian varieties (MA 5115)

As usual, we work over an algebraically closed field k.

Exercise 1. On the last exercise sheet, we studied the morphism $\phi_{\mathscr{L}}\colon A(k) \to \operatorname{Pic} A, x \mapsto t_x^*\mathscr{L} \otimes \mathscr{L}^{-1}$. In this exercise, we give a more explicit description in the case of a onedimensional Abelian varietiy E (i.e. elliptic curves). For this recall that the degree of a divisor $D = \sum n_i[x_i] \in Z^1(E)$ is defined as $\deg D = \sum n_i$.

(a) Let D be a divisor of degree d. Show that D is equivalent to a divisor of the form $[x]+(d-1)\cdot[0]$ with $x \in E(k)$. Prove that x is uniquely determined if $d \neq 0$.

(b*) Can you show that x is uniquely determined also for $d = 0$?

(c) Deduce that for $\mathscr{L} = \mathscr{O}([0])$, the map $\phi_{\mathscr{L}}$ defines an isomorphism $E(k) \xrightarrow{\sim} \operatorname{Pic}^0(E)$ (the latter consisting of line bundles $\mathscr{O}(D)$ with $\deg D = 0$).

(d) Deduce from the statement above that $\phi_{\mathscr{O}(D)}$ is the zero map if and only if $\deg D = 0$.

Exercise 2 (Pullback of divisors)**.** Let $\varphi\colon X \to Y$ be a dominant morphism of irreducible smooth varieties (in particular we get a morphism on fields of rational functions $\varphi^*\colon k(Y) \to k(X)$). Recall that the pullback defines a group homomorphism

$$\varphi^*\colon \operatorname{Pic}(Y) \to \operatorname{Pic}(X), \mathscr{L} \mapsto \varphi^*\mathscr{L}.$$

(a) Give a (formal) definition of the pull-back $\varphi^*(D)$ of a divisor $D \in Z^1(Y)$ using the forllowing (informal) description of $\varphi^*(D)$. “After passing to an open cover, we may assume $D = \operatorname{div}(\mathrm{f})$ for $f \in K(Y)$. Then define $\varphi^*D \coloneqq \operatorname{div}(\varphi^*(\mathrm{f}))$.”

(b) Now assume that X and Y are proper. Hence we know that $\operatorname{Cl}(X) \cong \operatorname{Pic}(X), D \mapsto \mathscr{O}(D)$. Show that $\mathscr{O}(\varphi^*D) \cong \varphi^*\mathscr{O}(D)$, i.e. the two notions of pullback are compatible.

Deadline: Monday, 21st December, 2020