Technische Universität München
Zentrum Mathematik

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Winter term 2020/21
Exercise sheet 5

Abelian varieties (MA 5115)

As usual, we work over an algebraically closed field k.

Exercise 1. Last week we claimed in the lecture that $\operatorname{Pic}\mathbb{P}^n \cong \mathbb{Z}$. We are now proving this statement (at least for $n = 1$) using divisors.

(a) Prove that every divisor of degree zero on $\mathbb{P}^1$ is a principal divisor.

(b) Prove that every principal divisor is of degree zero.

(c) Conclude that $\operatorname{Pic}\mathbb{P}^1 \cong \mathbb{Z}$.

(d*) can you repeat the proof for $n > 1$? (Hint: Every prime divisor is the vanishing set of a homogenous height one prime ideal $\mathfrak{p} \subset k[x_0, \ldots, x_n]$, which is principal by Nagata).

Exercise 2. Let E be an Abelian variety of dimension 1 (i.e. an elliptic curve) and let $O \in E(k)$ be the unit element. We will use the results from the lecture to construct an embedding of E into the projective space (in fact, we are constructing the Weierstraß embedding). We will later generalize this construction to Abelian varieties of arbitrary dimension.

Let $D_0 = 3 \cdot [O]$ and denote the corresponding complete linear system $\mathfrak{d}$.

(a) Show that for $a_1, a_2, a_3 \in E(k)$ such that $a_1 + a_2 + a_3 = 0$, we have $[a_1] + [a_2] + [a_3] \in \mathfrak{d}$.

(b) Conclude that $3D_0$ is very ample and hence defines a closed embedding into projective space.

Deadline: Monday, 7th December, 2020