Sheet 1

1.

(1) $(1+r)^{-1} = \sum_{i=1}^{\infty}(-r)^i$.

(2)

$(a) \Rightarrow (b)$ N is the intersection of all prime ideals, so here equal to the only prime ideal. It is also maximal, hence the claim.

$(b) \Rightarrow (c)$ N must be maximal, as every proper ideal is contained in $R \backslash R^\times$.

$(c) \Rightarrow (a)$ N must be maximal. As it is contained in any prime ideal it is the only one.

2. Just follow the outline of the proof from the lecture, to show that the map is well-defined and bijective.

3.

(1) Induction over $\deg(f)$ for both implications. $\deg(f) = 0$ is clear. Otherwise: $f = c_n X^n + g$ with $\deg(g) < n$. By induction $f = g \cdot (1 - (-c_n g^{-1} X^n))$ and thus

$$f^{-1} = g^{-1} \cdot \sum_{i=0}^{\infty} (-c_n g^{-1} X^n)^i.$$

This is also the inverse of f in the ring of formal power series. The necessary condition for it to be a polynomial is c_n nilpotent (as g^{-1} and X are not).

(2) f nilpotent. Then c_0 is nilpotent and hence also

$$f - c_0 = X \cdot \left(c_1 + ... + c_n X^{n-1}\right).$$

Inductively, all c_i are nilpotent.
For the other implication ww write again $f = c_n X^n + g$ with $\deg(g) < n$. Inductively we have $l_n, l_g \in \mathbb{N}$ s.t. $c_n^{l_n} = 0 = g^{l_g}$. Use the binomial theorem to show

$$f^{l_n + l_g - 1} = 0.$$

(3) Let $g = \sum_{i=0}^{m} b_i X^i \in R[X]$ with $gf = 0$ and $b_m \neq 0$. Further choose g such that m is minimal. Now assume $m \geq 1$. Then set

$$l := \max\{j \in \mathbb{N} \mid c_j g \neq 0\}.$$

Now we have

$$0 = fg = (c_0 + ... + c_n X^n) g = (c_0 + ... + c_l X^l) g.$$

Hence $c_l b_m = 0$. This implies $0 \leq \deg(c_l g) < m$. But $(c_l g) f = c_l (gf) = 0$ which contradicts the minimality of g.
The other implication is trivial.

4.

(1) Just verify the definition of a ring.

(2) For each $x \in I$ we have the map

$$\begin{aligned} R &\to \mathbb{R}, \\ f &\mapsto f(x). \end{aligned}$$

It is a surjectve ring homomorphism with kernel M_x. Hence $R/M_x \cong \mathbb{R}$ and so M_x is maximal.

(3) No. The interval I is of the form $I = (a, b)$ with $a, b \in \mathbb{R} \cup \{\pm\infty\}$. Now consider the set

$$A := \{f \in R \mid \exists c \in \mathbb{R} \text{ s.t. } supp(f) = (a, c]\}.$$

Then obviously $A \not\subseteq R^\times$. Furthermore, $A \not\subseteq M_x$ for all $x \in I$, since for all $x \in I$ we find $x > c \in I$ and $f \in R$ with support $(a, c]$. Hence A must be contained in another maximal ideal.

(4) No. The 1-function has no compact support.