Sheet 2

1. (1) First equality:

$$\begin{aligned} r \in ((I : J) : K) &\Leftrightarrow rK \subseteq (I : J) \\ &\Leftrightarrow rKJ \subseteq I \Leftrightarrow r \in (I : JK). \end{aligned}$$

   Second equality follows from the first.

   (2)

$$\begin{aligned} r \in (I : J + K) &\Leftrightarrow rJ + rK \subseteq I \\ &\Leftrightarrow rJ, rK \subseteq I \Leftrightarrow r \in (I : J) \cap (I : K). \end{aligned}$$

   (3) First,

$$r \in \sqrt{IJ} \Rightarrow \exists n \in \mathbb{N}_+ : r^n \in IJ \subseteq I \cap J \Rightarrow r \in \sqrt{I \cap J}.$$

   Second,

$$r \in \sqrt{I \cap J} \Rightarrow \exists n \in \mathbb{N}_+ : r^n \in I \cap J \Rightarrow \exists n \in \mathbb{N}_+ : r^n \in I \wedge r^n \in J \Rightarrow r \in \sqrt{I} \cap \sqrt{J}.$$

   Third,

$$r \in \sqrt{I} \cap \sqrt{J} \Rightarrow \exists n, m \in \mathbb{N}_+ : r^n \in I \wedge r^m \in J \Rightarrow r^{n+m} \in IJ.$$

   (4) First,

$$r \in \sqrt{P^n} \Rightarrow \exists m \in \mathbb{N}_+ : r^m \in P^n \subseteq P \overset{P \text{ prime}}{\Rightarrow} r \in P.$$

   Second,

$$r \in P \Rightarrow r^n \in P^n \Rightarrow r \in \sqrt{P^n}.$$

2. Let Σ be the set of prime ideals containing I. It is partially ordered by inclusion. Let $T \subset \Sigma$ be a chain. Set

$$J := \bigcap_{J_\alpha \in T} J_\alpha.$$

   This is an ideal containing I. It is also prime: Let $x, y \in R \backslash J$. Then there exists α, β such that $x \notin J_\alpha, y \notin J_\beta$. W.l.o.g. $J_\alpha \subset J_\beta$, so $x, y \notin J_\alpha$. Hence $xy \notin J_\alpha$ since J_α is prime and thus $xy \notin J$.
   By Zorn's Lemma a minimal prime ideal containing I exists.
   By applying this to $I = (0)$ we get that a minimal prime ideal in R exists.

3. (1)

   (a)⇒(b) Let $U \subseteq \mathbb{R}$ be open and $x \in f^{-1}(U)$. Then there exists $\varepsilon > 0$ such that the open ball $B_\varepsilon(f(x))$ is fully contained in U, as it is open. By (a) we find $\delta > 0$ such that $B_\delta(x) \subset f^{-1}(U)$. Hence it is open.

(b)$\Rightarrow$(a) Let $x \in \mathbb{R}$ and $\varepsilon > 0$. Then $B_\varepsilon(f(x))$ is open and by (b) its preimage $f^{-1}(B_\varepsilon(f(x)))$ is open, too, and contains x. Whence we find a $\delta > 0$ such that $B_\delta(x) \subset f^{-1}(B_\varepsilon(f(x)))$.

(2) Consider the map

$$g : Spec(S) \to Spec(R), P \mapsto f^{-1}(P).$$

Let $I \subset R$ be an ideal. Then $V_R(I)$ is closed. We show that $g^{-1}(V_R(I))$ is closed in $Spec(S)$:

$$\begin{aligned} g^{-1}(V_R(I)) &= \{Q \in Spec(S) \mid g(Q) \in V_R(I)\} \\ &= \{Q \in Spec(S) \mid I \subseteq g(Q)\} \\ &= \{Q \in Spec(S) \mid I \subseteq f^{-1}(Q)\} \\ &= \{Q \in Spec(S) \mid f(I) \subseteq Q\} \\ &= V_S(f(I)). \end{aligned}$$

4. (1) Set $d := \frac{|x-y|}{2}$ and $U_x := B_d(x)$, $U_y := B_d(y)$.

   (2) As R is an integral domain, (0) is a prime ideal. For all $x, y \in Spec(R)$ we have $(0) \subset x, y$. Thus, by definition of the Zariski topology, (0) lies in no closed proper subset of $Spec(R)$ and hence in every non-empty open subset. In particular, (0) lies in U_x and U_y.
   More general: In any topological space, any dense subset meets any non-empty open subset. So if a generic point exists, it lies in every non-empty open subset.

   (3) No. Choose $y = (0)$. Then $y \in U_x$. Compare (2).

So the conclusion is, that the Zariski topology is very coarse and the open sets are very big.