Sheet 3

1. (1) First, $\mathbb{R}$ is not irreducible, as $(0,1)$ and $(1,2)$ do not intersect. Second, a discrete space X is irreducible if and only if $|X| = 1$, since all one point sets are open. Third, a chaotic space X is irreducible because X is the only non-empty open set.

   (2) First we observe the following. Let $I, J \subseteq R$ be radical ideals. Then

$$\begin{aligned}
&\emptyset = V(I)^c \cap V(J)^c = (V(I) \cup V(J))^c = V(IJ)^c \\
\Leftrightarrow & V(IJ) = Spec(R) = V(Nil(R)) \\
\Leftrightarrow & IJ \subseteq Nil(R).
\end{aligned}$$

   Now first assume that $Nil(R)$ is prime. Let $V(I)^c$ and $V(J)^c$ be disjoint open sets. By the above observation, $IJ \subseteq Nil(R)$. And since $Nil(R)$ is prime, $I \subseteq Nil(R)$ or $J \subseteq Nil(R)$. I.e. $V(I)^c = \emptyset$ or $V(J)^c = \emptyset$, so $Spec(R)$ is irreducible.
   Now assume that $Nil(R)$ is not prime. We find $a, b \in R \setminus Nil(R)$ with $ab \in Nil(R)$. By the above observation the non-empty open sets $D(a)$ and $D(b)$ are disjoint. Thus $Spec(R)$ is not irreducible.

2. Since $f_3(i, 0) = 0$ we can set $M = (x - i, y) \subset \mathbb{C}[x, y]$. Now observe that both $\mathbb{R}[x, y]$ and $\mathbb{C}[x, y]$ are $\mathbb{R}$-algebras and $\mathbb{C}[x, y] = \mathbb{R}[i, x, y]$ is finitely generated. Hence, the preimage $(x^2 + 1, y)$ of M under the inclusion $\mathbb{R}[x, y] \hookrightarrow \mathbb{C}[x, y]$ is maximal in $\mathbb{R}[x, y]$. Choose $P = (x^2 + 1, y)$.

3. (1) First, $A = k[X]/\left(X^2\right)$ is not integral and no field. Second, $A = k[X]$ is integral, but not algebraic over k and no field.

   (2) $K = k(X)$ is contained in the integral domain $R = k(X) = K$, which is not a finitely generated k-algebra. And K is not algebraic over k.

4. (1) $R = k[[t]]$, $Quot(R) = k((t)) = k((t))[t^{-1}]$

   (2) $R = \mathbb{Z}$, $Quot(R) = \mathbb{Q}$

   (3) $R = \mathbb{Z}/6\mathbb{Z}$, $\text{Spec}(R) = \{(2), (3)\}$

   (4) $R = k[x, y]/(xy)$. Then $\text{Spec}(R)$ is connected, but has two irreducible components $V(x)$, $V(y) \neq \emptyset$ intersecting in the point with coordinates $x = 0$, $y = 0$ and $V(x) \cup V(y) = \text{Spec}(R)$.