Sheet 4

1. (1) Let R be Jacobson and $\mathfrak{p} \in \operatorname{Spec}(R)$. Then

$$\mathfrak{p} = \bigcap_{\mathfrak{p} \subseteq \mathfrak{m} \text{ maximal}} \mathfrak{m}.$$

Assume that every prime ideal is the intersection of **some** maximal ideals—not necessarily all containing that prime. Let I be any ideal. Then

$$\sqrt{I} = \bigcap_{I \subseteq \mathfrak{p} \text{ prime}} \mathfrak{p} = \bigcap_{I \subseteq \mathfrak{p} \text{ prime}} \left(\bigcap_{\mathfrak{m}_{\mathfrak{p}} \text{ maximal}} \mathfrak{m} \right) = \bigcap_{I \subseteq \mathfrak{m} \text{ maximal}} .$$

Observe that we get **all** maximal ideals containing I in the last equality, even so the prime ideals $\mathfrak{p}$ may **not** be the intersection of **all** maximal ideals containing them.

(2) Every prime ideal is maximal. Aplly (1).

(3) $k[[x]]$ is an integral domain and local with maximal ideal (x). So (0) is not an intersection of maximal ideals.

2. There is a k-algebra homomorphism $f\colon k[x,y] \to k[t]$, mapping x to t^3 and y to t^2. Then f factors through $k[x,y] \twoheadrightarrow R$, which induces an embedding $R \xrightarrow{\sim} k[t^2,t^3] \subseteq k[t]$. This induces also the map $\operatorname{Spec} k[t] \to \operatorname{Spec} R$. The ring $k[t^2,t^3]$ is not a principal ideal domain, as (t^2,t^3) is not a principal ideal, whereas $k[t]$ is principal ideal domain. Thus R cannot be isomorphic to $k[t]$.

3. Let $k[X] = k[X_1,\dots,X_n]/I(X)$. We have the natural map

$$\psi : X \to \operatorname{Hom}_k(k[X],k),$$
$$a = (a_1,\dots,a_n) \mapsto \varphi_a : \begin{pmatrix} k[X] & \to & k, \\ f + I(X) & \mapsto & f(a_1,\dots,a_n). \end{pmatrix}$$

Then ψ is well-defined by the definition of $I(X)$.
Let $a,b \in X$ with $\varphi_a = \varphi_b$. But then $a_i = \varphi_a(X_i + I(X)) = \varphi_b(X_i + I(X)) = b_i$ for all i. So $a = b$ and thus ψ is injective.
Now let $\varphi \in \operatorname{Hom}_k(k[X],k)$. It is determined by $\varphi(X_i + I(X))$. So set $a := (\varphi(X_1 + I(X)),\dots,\varphi(X_n + I(X)))$. Then $\varphi = \varphi_a$ and so ψ is surjective.

4. (1) Every homomorphism of R-modules is determined by the values at a basis (if existent). So we get the canonical map

$$M \to \operatorname{Hom}_R(R,M),$$
$$m \mapsto \begin{pmatrix} R & \to & M \\ 1 & \mapsto & m \end{pmatrix}.$$

It is now easy to see, that this is a bijective homomorphism of R-modules.

(2) We have the natural maps

$$f_* : \mathrm{Hom}_R(M, N) \to \mathrm{Hom}_R(M', N),$$
$$\varphi \mapsto \varphi \circ f$$

and

$$g^* : \mathrm{Hom}_R(M, N) \to \mathrm{Hom}_R(M, N'),$$
$$\psi \mapsto g \circ \psi.$$

Again it is easy to check that these are R-module homomorphisms.

(3) If $n = m$ then the claim is obviously true. So let $\varphi : R^n \to R^m$ be an isomorphism of R-modules. Choose a maximal ideal $\mathfrak{m} \subset R$. In the lecture we have seen that $R/\mathfrak{m} \otimes_R R^n \cong (R/\mathfrak{m})^n$ and $R/\mathfrak{m} \otimes_R R^m \cong (R/\mathfrak{m})^m$. Now check that

$$id \otimes \varphi : R/\mathfrak{m} \otimes_R R^n \to R/\mathfrak{m} \otimes_R R^m,$$
$$(r + \mathfrak{m}) \otimes x \mapsto (r + \mathfrak{m}) \otimes \varphi(x)$$

is an isomorphism of $R/\mathfrak{m}$-modules (i.e. vector spaces). We deduce that $(R/\mathfrak{m})^n \cong (R/\mathfrak{m})^m$ as $R/\mathfrak{m}$-vector spaces and hence $m = n$.