Sheet 5

1. (1) Consider the sequences

$$M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0 \tag{$\star$}$$

and

$$0 \to \operatorname{Hom}_R(M'', N) \xrightarrow{g_*} \operatorname{Hom}_R(M, N) \xrightarrow{f_*} \operatorname{Hom}_R(M', N). \tag{$\star\star$}$$

Now assume that $(\star)$ is exact.
Let $\varphi \in \ker(g_*)$, i.e. $\varphi \circ g = 0$. As g is surjective, we get $\varphi = 0$. So g_* is injective.
Let $\varphi \in \operatorname{Hom}_R(M'', N)$. Then $(f_*(g_*(\varphi)) = \varphi \circ g \circ f = 0$, since $\operatorname{im}(f) = \ker(g)$. So we get $\operatorname{im}(g_*) \subseteq \ker(f_*)$.
Let $\varphi \in \ker(f_*)$, i.e. $\varphi \circ f = 0$. This means $\ker(g) = \operatorname{im}(f) \subseteq \ker(\varphi)$. As g is surjective we have $M'' \cong M/\ker(g)$. Thus we can define $\psi : M'' \cong M/\ker(g) \to N$ by $\psi(m + \ker(g)) := \varphi(m)$. This is well-defined because of $\ker(g) \subseteq \ker(\varphi)$. We clearly have $\varphi = \psi \circ g$, so $\ker(f_*) \subseteq \operatorname{im}(g_*)$.

Now assume that $(\star\star)$ is exact.
Let $\varphi_1, \varphi_2 \in \operatorname{Hom}_R(M'', N)$ for any choice of N. Then the injectivity of g_* says

$$\varphi_1 \circ g = \varphi_2 \circ g \Rightarrow \varphi_1 = \varphi_2.$$

But this just means that g is right-cancellative, hence surjective.
Choose $N = M''$ and $id \in \operatorname{Hom}_R(M'', M'')$. As $\ker(f_*) = \operatorname{im}(g_*)$ we deduce $0 = f_*(g_*(id)) = g \circ f$, thus $\operatorname{im}(f) \subseteq \ker(g)$.
Choose $N = M/\operatorname{im}(f)$ and $\pi \in \operatorname{Hom}_R(M, M/\operatorname{im}(f))$ the canonical projection. Then $\pi \circ f = 0$, so $\pi \in \ker(f_*) = \operatorname{im}(g_*)$ So we find $\varphi \in \operatorname{Hom}_R(M'', M/\operatorname{im}(f))$ with $\pi = \varphi \circ g$. Now let $m \in \ker(g)$. We get $0 = \varphi(0) = \varphi(g(m)) = \pi(m)$. So $m \in \ker(\pi) = \operatorname{im}(f)$.

(2) In general, when $M' \subseteq M$, the induced map $\operatorname{Hom}_R(M, N) \to \operatorname{Hom}_R(M', N)$ is just the restriction $\varphi \mapsto \varphi|_{M'}$.
Choose $R = M = N = \mathbb{Z}$ and $M' = 2\mathbb{Z}$. Then $\operatorname{Hom}_\mathbb{Z}(\mathbb{Z}, \mathbb{Z}) \to \operatorname{Hom}_\mathbb{Z}(2\mathbb{Z}, \mathbb{Z})$ is not surjective, since the $\mathbb{Z}$-linear map

$$2\mathbb{Z} \to \mathbb{Z}, \qquad 2 \mapsto 1$$

is not induced by a $\mathbb{Z}$-linear map $\mathbb{Z} \to \mathbb{Z}$.

2. To prove (1) and (2), we show the following general result:
Let R be a ring and $I, J \subseteq R$ be ideals. Then $R/I \otimes_R R/J \cong /(I + J)$.
Proof: We have the bilinear map

$$R/I \times R/J \to R/(I + J), \qquad (\bar{r}, \bar{s}) \mapsto \overline{rs}.$$

(Check that it is well-defined!)
The universal property of the tensor product gives an R-linear map

$$\phi : R/I \otimes_R R/J \to R/(I+J), \qquad \bar{r} \otimes \bar{s} \mapsto \overline{rs}.$$

Now we make the following observation. Let $\sum \overline{r_i} \otimes \overline{s_i} \in R/I \otimes_R R/J$ be arbitrary. Then

$$\begin{aligned}\sum \overline{r_i} \otimes \overline{s_i} &= \sum r_i \cdot (\bar{1} \otimes \overline{s_i}) \\ &= \sum (r_i s_i \cdot (\bar{1} \otimes \bar{1})) \\ &= \left(\sum r_i s_i\right) \cdot (\bar{1} \otimes \bar{1})\end{aligned}$$

So every element of $R/I \otimes_R R/J$ is of the form $r \cdot (\bar{1} \otimes \bar{1})$ for some $r \in R$. Thus the above map is given by $\phi(r \cdot (\bar{1} \otimes \bar{1})) = \bar{r}$. It is obviously surjective. To see that is is injective, too, use that for all $i \in I, j \in J$ we have

$$(i+j) \cdot (\bar{1} \otimes \bar{1}) = i \cdot (\bar{1} \otimes \bar{1}) + j \cdot (\bar{1} \otimes \bar{1}) = \bar{i} \otimes \bar{1} + \bar{1} \otimes \bar{j} = \bar{0} \otimes \bar{1} + \bar{1} \otimes \bar{0} = 0.$$

Alternatively, we can aplly exercise 3. and get

$$\begin{aligned}R/I \otimes_R R/J &\overset{3.}{\cong} (R/I)/(J(R/I)) \\ &= (R/I)/(J/(I \cap J)) \\ &= (R/I)/(I+J/I) \\ &= R/(I+J).\end{aligned}$$

(2) Applying the above result we get $(\mathbb{Z}/m\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/\gcd(m,n)\mathbb{Z}$.

(1) Follows from (2).

(3) We had this in the lecture for free modules of rank m and n over arbitrary rings R. Aplly for $R = k$.

3. We have the exact sequence

$$0 \to I \to R \to R/I \to 0.$$

Tensoring with M yields the exact sequence

$$I \otimes_R M \to M \to R/I \otimes_R M \to 0.$$

We have $I \otimes_R M \cong IM$ and $R/I \otimes_R M \cong M/\ker(M \to R/I \otimes_R M)$. Further more $\ker(M \to R/I \otimes_R M) = \operatorname{im}(I \otimes_R M \to M) = IM$.

Alternatively, we can consider the bilinear map

$$R/I \times M \to M/IM, \qquad (\bar{r}, m) \mapsto \overline{rm}.$$

We get an R-linear map

$$\phi : R/I \otimes_R M \to M/IM, \qquad \overline{r} \otimes m \mapsto \overline{rm}.$$

Now let $\sum \overline{r_i} \otimes m_i \in R/I \otimes_R M$ be arbitrary. Then

$$\begin{aligned} \sum \overline{r_i} \otimes m_i &= \sum r_i(\overline{1} \otimes m_i) \\ &= \sum (\overline{1} \otimes (r_i m_i)) \\ &= \overline{1} \otimes \left(\sum r_i m_i\right). \end{aligned}$$

Thus every element of $R/I \otimes_R M$ is of the form $\overline{1} \otimes m$ for some $m \in M$. Now we can again easily show that ϕ is bijective.

4. Just follow the hint. To show the implictation

$$M \otimes_R N = 0 \Rightarrow M_k \otimes_k N_k = 0$$

it is helpful to use the following cancellation law:
Let R be a ring, M an R-module, A an R-algebra and N an A-module. Then

$$(M \otimes_R A) \otimes_A N \cong M \otimes_R N,$$

as A-modules.
Proof: Consider the (A-)bilinear map

$$(M \otimes_R A) \times N \to M \otimes_R N, \qquad ((m \otimes a), n) \mapsto m \otimes (an).$$

It induces the A-linear map

$$\phi : (M \otimes_R A) \otimes_A N \to M \otimes_R N, \qquad (m \otimes a) \otimes n \mapsto m \otimes (an).$$

Next consider the (R-)bilinear map

$$M \times N \to (M \otimes_R A) \otimes_A N, \qquad (m, n) \mapsto (m \otimes 1) \otimes n.$$

It induces the R-linear map

$$\psi : M \otimes_R N \to (M \otimes_R A) \otimes_A N, \qquad m \otimes n \mapsto (m \otimes 1) \otimes n.$$

ψ is obviously A-linear. Now it is easy to see that ϕ and ψ are inverse to each other.