Sheet 6

1. Let N be a S-module. With the cancelation law from the last sheet's solution we know that
$$(M \otimes_R S) \otimes_S N \cong M \otimes_R N.$$
With this it is easy to see that $M \otimes_R S$ is flat if M is flat.

2. (1) Consider the short exact sequence
$$0 \to \mathbb{Z} \overset{\cdot m}{\to} \mathbb{Z} \overset{\pi}{\to} \mathbb{Z}/m\mathbb{Z} \to 0$$
with multiplying with m first and then projecting. As $\mathbb{Z}$ is free (hence projective), this is a projective resolution for $\mathbb{Z}/m\mathbb{Z}$.

   (2) First,
$$\mathrm{Tor}_0^{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, N) = \mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} N \cong N/mN.$$
Second,
$$\begin{aligned}\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, N) &= \ker(\mathbb{Z} \otimes_{\mathbb{Z}} N \overset{\cdot m \otimes \mathrm{id}}{\to} \mathbb{Z} \otimes \mathbb{Z}N)/\mathrm{im}(0 \otimes_{\mathbb{Z}} N \to \mathbb{Z} \otimes_{\mathbb{Z}} N) \\ &= \ker(\mathbb{Z} \otimes_{\mathbb{Z}} N \overset{\cdot m \otimes \mathrm{id}}{\to} \mathbb{Z} \otimes \mathbb{Z}N).\end{aligned}$$
Since $\mathbb{Z} \otimes_{\mathbb{Z}} N \cong N$, we have furthermore
$$\ker(\mathbb{Z} \otimes_{\mathbb{Z}} N \overset{\cdot m \otimes \mathrm{id}}{\to} \mathbb{Z} \otimes N) \cong \ker(N \overset{\cdot m}{\to} N) = \{n \in N \mid mn = 0\}.$$
Third, $\mathrm{Tor}_i^{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, N) = 0$ for $i \geq 2$ is clear.

3. The proofs of both, the snake lemma and the 5-lemma, involve standard diagram chase arguments, and can be found in many introductory books on commutative algebra (or in the internet).

4. As V is finite dimensional, its dual can be given by $V \cong V^\vee$ via $v \mapsto v^T$. We have the bilinear map
$$\begin{aligned} V^\vee \times W &\to \mathrm{Hom}_k(V, W) \\ (\varphi^T, w) &\mapsto \begin{pmatrix} V & \to & W \\ v & \mapsto & (\varphi^T v)w \end{pmatrix}\end{aligned}$$
which induces the k-linear map
$$\begin{aligned} \psi : V^\vee \otimes_k W &\to \mathrm{Hom}_k(V, W) \\ \varphi^T \otimes w &\mapsto \begin{pmatrix} V & \to & W \\ v & \mapsto & (\varphi^T v)w \end{pmatrix}\end{aligned}$$

Observe that $(\varphi^T v)w = w(\varphi^T v) = (w\varphi^T)v$. So $w\varphi^T$ is the transformation matrix for the linear map $\psi(\varphi^T \otimes w) : V \to W$ and has rank 1.
Now let $\dim V = n, \dim W = m$ and $A \in k^{m\times n}$ be the transformation matrix of any linear map $l_A : V \to W$. Denote its columns by $a_1, ..., a_n \in k^m \cong W$ and let $e_1^T, ..., e_n^T$ be the standard basis of $V^\vee$. Then

$$\begin{aligned} A &= (a_1 \mid ... \mid a_n) \\ &= (a_1 \mid 0 \mid ... \mid 0) + ... + (0 \mid ... \mid 0 \mid a_n) \\ &= a_1 \cdot e_1^T + ... + a_n \cdot e_n^T \end{aligned}$$

This implies that $l_A = \psi\left(\sum_{i=1}^n e_i^T \otimes a_i\right)$, so ψ is surjective. Moreover we have

$$\begin{aligned} \dim(V^\vee \otimes_k W) &= \dim(V^\vee)\dim(W) \\ &= \dim(V)\dim(W) \\ &= \dim(\mathrm{Hom}_k(V, W)), \end{aligned}$$

so ψ is also bijective.