Sheet 7

1. Just copy the proof from the lecture.

2. We have the canonical homomorphism $f : R \to R_{\mathfrak{p}}$. Consider the maps

$$\begin{array}{ccc} \operatorname{Spec}(R_{\mathfrak{p}}) & \longleftrightarrow & \{\mathfrak{q} \in \operatorname{Spec}(R) \mid \mathfrak{q} \subseteq \mathfrak{p}\} \\ \mathfrak{I} & \longmapsto & f^{-1}(\mathfrak{I}) \stackrel{!}{=} \{r \in R \mid \frac{r}{1} \in \mathfrak{I}\} \\ \mathfrak{q}_{\mathfrak{p}} = \{\frac{x}{s} \mid x \in \mathfrak{q}, s \notin \mathfrak{p}\} & \leftarrow & \mathfrak{q} \end{array}$$

   It is easy to see that these maps are well-definied and inverse to each other. The given description for $f^{-1}(\mathfrak{I})$ may be helpful and is also easy to prove.

   - The set $\{\mathfrak{q} \in \operatorname{Spec}(R) \mid \mathfrak{p} \subseteq \mathfrak{q}\}$ from (1) is just $V(\mathfrak{p})$—so it is Zariski-closed.
   - If, for example, $(R, \mathfrak{m})$ is local then $\{\mathfrak{q} \in \operatorname{Spec}(R) \mid \mathfrak{q} \subseteq \mathfrak{m}\} = \operatorname{Spec}(R)$, i.e. Zariski-open <u>and</u> -closed. This also shows that localizing local rings at the maximal ideal is boring. But in general the set $\{\mathfrak{q} \in \operatorname{Spec}(R) \mid \mathfrak{q} \subseteq \mathfrak{p}\}$ from (2) is neither open nor closed.
     Take for example $R = \mathbb{Z}$ and $\mathfrak{p} = (0)$. In $\operatorname{Spec}(\mathbb{Z})$ every non-empty open subset contains a maximal ideal. And every non-empty open subset contains infinitly many prime ideals. So the set $\{\mathfrak{q} \in \operatorname{Spec}(\mathbb{Z}) \mid \mathfrak{q} \subseteq \mathfrak{p}\} = \{(0)\}$ is neither Zariski-closed nor -open.

3. We give a description of $\operatorname{Spec} R$. We have an injection $R \cong k[t^2, t^3] \subseteq k[t]$. This induces an isomorphism between the localizations of R at the multiplicative subset $S_1 = \{t^i : i \geq 0; i \neq 1\}$ and of $k[t]$ at the multiplicative subset $S_2 = \{t^i : i \geq 0\}$:

$$S_1^{-1}R \cong S_2 k[t] \cong k[t, t^{-1}].$$

   Let $U = D(t^2)$ be an open subset of $\operatorname{Spec} R$. As in the exercise 2, we see that there is a bijection $U = \operatorname{Spec} S_1^{-1}R$. Thus the above shows that the open subset $U \subseteq \operatorname{Spec} R$ is isomorphic to $\operatorname{Spec} k[t, t^{-1}]$, which structure we already know (at least if k is algebraically closed). It remains to give a description of $\operatorname{Spec}(R) \smallsetminus U = V(t^2)$. But this set consists of exactly one point, corresponding to the maximal ideal $(t^2, t^3) \subseteq R$.

4. a) For $S = R^*$, one sees easily (either directly, or by using the universal mapping property) that $S^{-1}R \cong R$.

   b) If S is the complement in R to the set of all zero-divisors, then S is multiplicative and a direct computation shows that $R \to S^{-1}R$ is injective. Let T be a multiplicative subset of R, which contains a zero-divisor x such that $xy = 0$ in R for some $y \neq 0$. Then under $R \to T^{-1}R$, y goes to 0. Thus: if $T \subseteq R$ is multiplicative and $R \to T^{-1}R$ is injective, then $T \subseteq S$. Hence S is the unique maximal subset of R with the reqired property.