Sheet 8

1. $S^{-1}R \cong k[x, x^{-1}]$ (note that $\frac{y}{1} = \frac{0}{1}$ in the localization, as $xy = 0$ in R). Further, $k[t]/t^2$ is a local ring, hence its localization at the (unique) prime ideal is still isomorphic to $k[t]/t^2$.

2. Ideals in R/I correspond to ideals in R containing I. This is in particular true for maximal ideals. So for a maximal ideal $I \subset \mathfrak{m} \subset R$ let $\tilde{\mathfrak{m}}$ be its counterpart in R/I. Furthermore M/IM is an R/I-module. We know that $M/IM = 0$ if and only if $(M/IM)_{\tilde{\mathfrak{m}}} = 0$ for all maximal ideals $I \subset \mathfrak{m} \subset R$. This is equivalent to $M_{\mathfrak{m}}/(IM)_{\mathfrak{m}} = 0$ for all maximal ideals $I \subset \mathfrak{m} \subset R$. But this now holds by assumption.

3. $x = 0$ in $R_{\mathfrak{p}}$ means that there is an $s \in R \backslash \mathfrak{p}$ such that $sx = 0$ in R. As $0 \in \mathfrak{p}$, $s \notin \mathfrak{p}$ and $\mathfrak{p}$ prime, we deduce $x \in \mathfrak{p}$. Thus x lies in the intersection of all prime ideals of R, i.e., $x = 0$ as R reduced.

4. (1): '$\Leftarrow$' is immediate. Conversly, assume $M_{\mathfrak{p}} = 0$ for all $\mathfrak{p} \in \mathrm{Spec}(R)$. Applying Exercise 2 above to $I = (0)$, we deduce $M = (0)\dot{M} = 0$. This shows '$\Rightarrow$'.

   (2): If M is generated by one element m, one checks that $M \cong R/\mathrm{Ann}(m)$, where $\mathrm{Ann}(m) = \{r \in R \colon rm = 0\}$, and hence $\mathrm{Supp}(M) = V(\mathrm{Ann}(m))$. Let $m_1, \ldots, m_r$ be a set of generators of M. We have

$$\mathrm{Supp}(M) = \bigcup_i \mathrm{Supp}(Rm_i) = \bigcup_i V(\mathrm{Ann}(m_i)) = V(\bigcap_i \mathrm{Ann}(m_i)).$$

   As m_i generate M, we see that $r \in R$ lies in $\mathrm{Ann}(M)$ if and only if $rm_i = 0$, i.e., if $r \in \bigcap_i \mathrm{Ann}(m_i)$.

   (3): this is clear from (2).