Sheet 9

1. $R = R[x]/(x)$. So if $R[x]$ is noetherian, the same holds for R.

2. (1) We have that $\ker(f) \subseteq \ker(f^2) \subseteq ...$ This terminates, so there exists $n \in \mathbb{N}$ with $\ker(f^n) = \ker(f^{n+1})$. Now let $x \in \ker(f)$. As f is surjective, we find $y \in M$ with $f^n(y) = x$. But then $f^{n+1}(y) = f(f^n(y)) = f(x) = 0$. Thus $y \in \ker(f^{n+1}) = \ker(f^n)$. But then $x = f^n(y) = 0$. So f is injective.

   (2) We have that $\mathrm{im}(f) \supseteq \mathrm{im}(f^2) \supseteq ...$ This terminates, so there exists $n \in \mathbb{N}$ with $\mathrm{im}(f^n) = \mathrm{im}(f^{n+1})$. So for every $x \in M$ we have $f^n(x) \in \mathrm{im}(f^n)$ and we find $y \in M$ with $f^n(x) = f^{n+1}(y) = f^n(f(y))$. Since f is injective we deduce $x = f(y)$.

3. Let $m_1, ..., m_n$ be a set of generators for M. Consider the map

$$\begin{aligned} f : R &\to M^n, \\ r &\mapsto (rm_1, ..., rm_n). \end{aligned}$$

Its kernel is precisely I. So R/I is ismorphic to a submodule of M^n. As M is noetherian, M^n is noetherian, too. So R/I is noetherian as an R-module. But this is equivaent to R/I being noetherian as a ring.

Replacing "noetherian" by "artinian" gives a false statement. Consider $R = \mathbb{Z}$, p a prime number and $M = \mathbb{Z}\left[\frac{1}{p}\right]/\mathbb{Z}$. Then M can be seen as the set of all fractions between 0 and 1 and with powers of p as denominators.
Claim: Every proper $\mathbb{Z}$-submodule (i.e. every subgroup) of M is generated by $\frac{1}{p^k}$ for an appropriate $k \in \mathbb{N}$.
Proof: Let $k \in \mathbb{N}$ and $a \in \mathbb{Z}$ with $0 \leq a \leq p^k$ and $(a, p) = 1$. Then—with the euclidean algorithm—we find $b \in \mathbb{Z}$ with $ab \equiv 1 \mod p^k$. This means that $b \cdot \frac{a}{p^k} = \frac{1}{p^k}$ in M.
Multiplying any $\frac{a}{p^k}$ by elements in $\mathbb{Z}$ makes the denominator at worst smaller (cancelling down). So every proper subgroup is generated by its element $\frac{1}{p^k}$ with k maximal. $\square$
So every decreasing chain of submodules if of the form

$$\left\langle \frac{1}{p^{k_1}} \right\rangle \supseteq \left\langle \frac{1}{p^{k_2}} \right\rangle \supseteq ...$$

with $k_1 \geq k_2 \geq ... \geq 0$. And this surely terminates.

4. (1) Let $A \subset X$. Then every open set of A is of the form $A \cap U$ with $U \subset X$ open. Assume that A is not noetherian, so we find a strictly increasing chain

$$A \cap U_1 \subsetneq A \cap U_2 \subsetneq ...$$

of open sets in A. But this gives a strictly increasing chain

$$U_1 \subsetneq U_1 \cup U_2 \subsetneq ...$$

of open sets in X.

(2) Let $\mathcal{U}$ be an open covering of X. Let Σ be the set of finite unions of elements of $\mathcal{U}$. As X is noetherian, Σ has a maximal element $U_1 \cup ... \cup U_n$. By its maximality it must be the whole space X.

(3) Let $\mathcal{U}$ be an open covering of $\mathrm{Spec}(R)$, w.l.o.g. consisting of principal open sets. I.e. $\mathcal{U} = \{D(f_i) \mid f_i \in R, i \in I\}$ for an index set I. The condition

$$\bigcup_{i \in I} D(f_i) = \mathrm{Spec}(R)$$

is equivalent to

$$(f_i \mid i \in I) = R.$$

So we find a **finite** subset $H \subset I$ such that

$$(f_i \mid i \in H) = R$$

and

$$\bigcup_{i \in H} D(f_i) = \mathrm{Spec}(R).$$

(4) **Ascending** chains of **open sets** $\mathrm{Spec}(R) \backslash V(J_i)$ in $\mathrm{Spec}(R)$ give **descending** chains of **closed sets** $V(J_i)$ in $\mathrm{Spec}(R)$. And these give **acending** chains of **ideals** J_i in R. The last terminate, so do the first.