Sheet 11

1. Let $s \in S'$. Then we find a monic $p \in f(S)[X]$ with $p(s) = 0$. Clearly, we can lift p to a monic polynomial in $(f(S) \otimes_R T)[X]$, also denoted by p, such that $p(s \otimes 1) = 0$. So all elements of the form $s \otimes 1 \in S' \otimes_R T$ are integral over $f(S) \otimes_R T$. Moreover, $1 \otimes t \in f(S) \otimes_R T \subset S' \otimes_R T$ is integral over $f(S) \otimes_R T$ for all $t \in T$. So all simple tensors are integral, therefore the whole ring.

2. (1) The realation is well-definied, as $x \in R$ if and only if $xr \in R$ for all $r \in R\times$.

   For all $x \in K^\times$ we have $v(x) \geq v(x)$ since $xx^{-1} \in R$.

   If $v(x) \geq v(y)$ and $v(y) \geq v(x)$, we have $xy^{-1} \in R$ and $yx^{-1} \in R$. So $xy^{-1} \in R^\times$, i.e. $v(x) = v(y)$,

   If $v(x) \geq v(y)$ and $v(y) \geq v(z)$, we have $xy^{-1} \in R$ and $yz^{-1} \in R$. So $xz^{-1} = xy^{-1}yz^{-1} \in R$ and thus $v(x) \geq v(z)$.

   As R is a valuation ring, we have $xy^{-1} \in R$ or $yx^{-1} \in R$ for all $x, y \in K^\times$. So $v(x) \geq v(y)$ or $v(y) \geq v(x)$.

   (2) Let $v(x) \geq v(y)$, so $xy^{-1} \in R$. But then we have for all $z \in K^\times$ that $xzz^{-1}y^{-1} \in R$, so $v(x)v(z) \geq v(y)v(z)$.

   (3) If $v(x) \geq v(y)$ $xy^{-1} \in R$, we have $(x+y)y^{-1} = xy^{-1} + 1 \in R$, so $v(x+y) \geq v(y)$.

   If $v(y) \geq v(x)$ $yx^{-1} \in R$, we have $(x+y)x^{-1} = 1 + yx^{-1} \in R$, so $v(x+y) \geq v(x)$.

   (4) First,
   $$R^\times = \{x \in K^\times \mid v(x) = v(1)\}.$$
   Second,
   $$R = \{x \in K^\times \mid v(x) \geq v(1)\}.$$
   Third,
   $$\mathfrak{m} = R \backslash R^\times = \{x \in K^\times \mid v(x) > v(1)\}.$$

3.
   - $\mathcal{O}_K$ is normal, as it is integrally closed in a field.
   - $\mathcal{O}_K$ is noetherian, as it is finitely generated over $\mathbb{Z}$ which is noetherian.
   - Let $\mathfrak{p} \subset \mathcal{O}_K$ be a prime ideal. Then $\mathfrak{p} \cap \mathbb{Z} = (p)$ is prime in $\mathbb{Z}$. If p is prime, then (p) is maximal and so is $\mathfrak{p}$, since both $\mathbb{Z}$ and $\mathcal{O}_K$ are integral domains. So we have to show: $\mathfrak{p} \neq (0)$, then $p \neq 0$.

     Let $x \in \mathfrak{p} \backslash \{0\}$. Then there is an integral equation over $\mathbb{Z}$
     $$x^n + r_{n-1}x^{n-1} + ... + r_1 x + r_0 = 0$$

with w.l.o.g. $r_0 \neq 0$ (otherwise cancel x on both sides). The first $n-1$ summands on the left hand side are in $\mathfrak{p}$ and the right hand side is also in $\mathfrak{p}$. So $r_0 \in \mathfrak{p}$ and, of course, $r_0 \in \mathbb{Z}$. So $0 \neq r_0 \in (p)$, i.e. $p \neq 0$.

- To see that $\mathcal{O}_K$ is indeed a finitely generated $\mathbb{Z}$-module we refer to any book about algebraic number theory, for example the standard book by Jürgen Neukirch.

4. A closed subset of $\mathrm{Spec}(S)$ is of the form $V(I) \cong \mathrm{Spec}(S/I)$ for some ideal I of S. It is enough to show that $\phi(V(I)) = V(f^{-1}(I))$. As $V(f^{-1}(I)) \cong \mathrm{Spec}(R/f^{-1}(I))$, it is enough to show that the map

$$\bar{\phi}\colon \mathrm{Spec}(S/I) \to \mathrm{Spec}(R/f^{-1}(I))$$

induced by $\bar{f}\colon R/f^{-1}(I) \hookrightarrow S/I$ is surjective. As S is intergal over R, also S/I is integral over $R/f^{-1}(I)$. Hence (by a part of the going-up theorem), it follows that $\bar{\phi}$ is surjective.