Sheet 12

1. (1.) Obviously, $0 \in H$, so H is non-empty. Let U be any open neighbourhood of 0. Then $-U$ is one, too, so $-H = H$. Let $x, y \in U$. Then $U - y$ is open and contains y. So $x \in U - y$, i.e. $x + y \in U$.

   (2.) Obviously, $K \subset \overline{K}$, so $\overline{K}$ is non-empty. Let U be any closed set containing K. Then $-U$ is closed and contains $-K = K$. So $-\overline{K} = \overline{K}$. Now first, let $x \in K$. Then $K = x + K \subset x + \overline{K}$. Since $x + \overline{K}$ is closed we have $\overline{K} \subset x + \overline{K}$ and $-x + \overline{K} \subset \overline{K}$, whence $K + \overline{K} \subset \overline{K}$. Now second, let $x \in \overline{K}$. Then $K + x \subset \overline{K}$ by the above computations. This implies $\overline{K} + x = \overline{K + x} \subseteq \overline{K}$. We deduce $\overline{K} + \overline{K} \subset \overline{K}$.

2. (1.) Let μ, ι denote the addition and inversion maps. Let $x \in \mathbb{Z}$ and $n \in \mathbb{N}$ be arbitrary. First,

$$\iota^{-1}(U_n(x)) = U_n(-x).$$

   Second,

$$\mu^{-1}(U_n(x)) = \bigcup_{y \in \mathbb{Z}} (U_n(y) \times U_n(x - y)).$$

   Both right hand sides are open.

   (2.) $(a_k)_{k \in \mathbb{N}}$ is Cauchy if and only if for all open neighbourhoods $U_N(0)$ of zero there exists an $n_0 \in \mathbb{N}$, such that for all $k, k' \geq n_0$ we have $a_k - a_{k'} \in U_N(0) = p^N\mathbb{Z}$. I.e. $p^N \mid a_k - a_{k'}$.

   (3.) First, using $k' = k + 1$ in part (2.), we see that the sequence $(a_k)_{k \in \mathbb{N}}$ is Cauchy if and only if for all $N \in \mathbb{N}$ there exists $n_0 \in \mathbb{N}$ such that for all $k \geq n_0$ we have $p^N \mid c_k$. This we can reformulate to $v_p(c_k) \to \infty$ in $\mathbb{N}$ (with ordinary metric topology) or to $c_k \to 0$ in $\mathbb{Z}$ with p-adic topology.

   Second, using the result for $(a_k)_{k \in \mathbb{N}}$, we see that $(a'_k)_{k \in \mathbb{N}}$ is always Cauchy. I.e. in the p-adic completion $\mathbb{Z}_p$ all power series in p converge.