# Global degree bounds and the transfer principle for invariants

Roger M. Bryant and Gregor Kemper

April 6, 2004

**Abstract**

The main purpose of this paper is to verify a conjecture of Derksen and Kemper concerned with the boundedness of rings of invariants. We shall make use of some results connected with the "transfer principle" in invariant theory. These results are dealt with in some generality because of their independent interest and potential for further applications.

## 1 Introduction

If $K$ is an infinite field and $V$ is a finite-dimensional $K$-space (that is, vector space over $K$) we write $K[V]$ for the $K$-algebra of polynomial functions on $V$, which by definition is the polynomial ring generated by any basis of the dual space $V^*$ of $V$. It is a graded algebra, $K[V] = \bigoplus_{i \geqslant 0} K[V]_i$. If $G$ is a group acting linearly on $V$, then $G$ also acts on $K[V]$ by left translations: for $\xi \in K[V]$ and $g \in G$, $g\xi$ is defined by $(g\xi)(v) = \xi\left(g^{-1}v\right)$ for all $v \in V$. The ring of invariants, $K[V]^G$, is also a graded algebra, $K[V]^G = \bigoplus_{i \geqslant 0} K[V]_i^G$. For any graded $K$-algebra, $A = \bigoplus_{i \geqslant 0} A_i$, we write

$$\beta(A) = \min\left\{d \in \mathbb{N} : A \text{ is generated by } A_0 \oplus \cdots \oplus A_d\right\},$$

where, by convention, the minimum over an empty set is $\infty$.

Now suppose that $G$ is a linear algebraic group over an algebraically closed field $K$. By a $G$-module we mean a finite-dimensional $K$-space $V$ with a linear $G$-action which is given by a morphism of varieties $G \times V \to V$. (By contrast, a $KG$-module is a $K$-space, not necessarily finite-dimensional, with a linear $G$-action which need not be given by a morphism.) We define $\beta(G)$ to be the element of $\mathbb{N} \cup \{\infty\}$ given by

$$\beta(G) = \sup\left\{\beta\left(K[V]^G\right) : V \text{ a } G\text{-module}\right\}.$$

We say that $G$ has a *global degree bound on invariants* if $\beta(G) < \infty$, that is, there exists an integer $m$ such that $\beta\left(K[V]^G\right) \leqslant m$ for every $G$-module $V$. Note that in [2] the shorter phrase "$G$ has a global degree bound" is used for the same property. Our main result is as follows.

**Theorem 1.1.** *Let $G$ be a linear algebraic group over an algebraically closed field $K$. Then the following conditions are equivalent:*

 (a) *$G$ has a global degree bound on invariants;*

*(b) G is finite and* $\mathrm{char}(K)$ *does not divide the order of* $G$.

This establishes Conjecture 2.3 of Derksen and Kemper [2]. In [2] the case $\mathrm{char}(K) = 0$ was settled. Our proof of Theorem 1.1 proceeds in the following steps. The implication "(b) $\Rightarrow$ (a)" is given by the Noether bound, which in characteristic zero or larger than $|G|$ goes back to Noether [9], and which was recently proved independently by Fleischmann [5] and Fogarty [6] to hold also if $\mathrm{char}(K) < |G|$ but $\mathrm{char}(K) \nmid |G|$.

In order to prove the converse implication "(a) $\Rightarrow$ (b)" we proceed case by case. If $G$ is finite with $|G|$ divisible by $\mathrm{char}(K)$, then $G$ does not have a global degree bound on invariants by results of Richman [10]. It remains to prove that if $G$ is infinite then $\beta(G) = \infty$. In Section 2 of this paper we deal with the case where the connected component $G^\circ$ is not unipotent. In particular the result holds for $\mathrm{SL}_2(K)$. Also in Section 2 we reduce the case where $G$ is infinite and $G^\circ$ is unipotent to the special case where $G$ is the additive group of $K$. Finally, in Section 3, this special case is deduced from the result for $\mathrm{SL}_2(K)$ by means of an isomorphism commonly known as "Roberts' isomorphism" (see Example 3.6).

The strategy of our proof is very similar to that in Derksen and Kemper [2]. The main result of [2] shows that if $G$ is infinite and $\mathrm{char}(K) = 0$ then $\beta(G) = \infty$. Here we establish the same result for arbitrary $K$.

The need to make sure that Roberts' isomorphism holds in arbitrary characteristic led us to a close study of results connected with the transfer principle (see Grosshans [7]), and thus to the reformulation of Roberts' isomorphism as a special case of a result which yields an isomorphism under much more general hypotheses (Corollary 3.5). We give two approaches to the verification of these hypotheses, one which is completely elementary and works for any infinite ground field (Theorem 3.2), and another (Theorem 3.4) which uses some non-elementary facts from the theory of algebraic groups, and therefore requires an algebraically closed ground field (or a scheme-theoretic setting). This material is presented in Section 3, where we also give some further references to the literature.

## 2   Global degree bounds

In this section, $G$ is a linear algebraic group over an algebraically closed field $K$, and $G^\circ$ is the connected component containing the identity. Note that there exists a faithful $G$-module (either by definition of the linearity of $G$ or, if "linear" is taken to mean "affine", by Humphreys [8, Theorem 8.6]). We begin by dealing with the case where $G^\circ$ is not unipotent. In the following proposition, only the implication "(c) $\Rightarrow$ (a)" is needed for the proof of Theorem 1.1. We include the other implications for the sake of completeness.

**Proposition 2.1.** *The following conditions are equivalent:*

(a) $G^\circ$ *is unipotent;*

(b) *there exist only finitely many isomorphism types of irreducible* $G$-*modules;*

(c) *there exists a faithful* $G$-*module* $U$ *such that there are only finitely many isomorphism types of irreducible* $G$-*modules occurring among the composition factors of the modules* $K[U]_i$ *for* $i \geqslant 0$.

*Proof.* We start by showing that (a) implies (b). Let $V$ be a non-zero $G$-module. Since $G^\circ$ is normal in $G$, the subspace $V^{G^\circ}$ of $G^\circ$-invariants is a $G$-submodule. By Humphreys [8, Theorem 17.5], (a) implies that $V^{G^\circ}$ is non-zero. Thus, if $V$ is irreducible, $V = V^{G^\circ}$ and so $V$ may be regarded as a $G/G^\circ$-module, yielding an irreducible $K(G/G^\circ)$-module. Non-isomorphic irreducible $G$-modules clearly yield non-isomorphic irreducible $K(G/G^\circ)$-modules. Since there are only finitely many isomorphism types of irreducible $K(G/G^\circ)$-modules, we obtain (b).

The implication "(b) $\Rightarrow$ (c)" is clear from the fact that there exists a faithful $G$-module, as observed above.

To prove that (c) implies (a), assume that $G^\circ$ is not unipotent. Let $T$ be a maximal torus of $G^\circ$. Then $T \neq 1$ (otherwise $G^\circ$ is nilpotent by [8, Proposition 21.4B] and so $G^\circ$ is unipotent by [8, Theorem 19.3]). We will show that for every faithful $G$-module $U$ infinitely many isomorphism types of $G$-modules occur among the composition factors of the $K[U]_i$. If two irreducible $G$-modules do not have the same composition factors when regarded as $T$-modules then they are not isomorphic. Thus it suffices to prove the result in the case $G = T$. Hence we may assume that $G = D_n(K)$ (a direct product of $n$ copies of the multiplicative group) for some $n \geqslant 1$. Thus (see [8, §16]) each $G$-module is the direct sum of 1-dimensional submodules, where each such submodule is determined up to isomorphism by its character $\alpha : G \to K \setminus \{0\}$. Since $U$ is faithful, $K[U]_1$ has a summand with character $\alpha$ where $\alpha \neq 1$. Hence $K[U]_i$ has a summand with character $\alpha^i$. Since the characters $1, \alpha, \alpha^2, \ldots$ are distinct, (c) cannot hold. Thus indeed (c) implies (a). $\square$

**Proposition 2.2.** *Suppose that $G^\circ$ is not unipotent. Then $\beta(G) = \infty$.*

*Proof.* Let $k$ be an arbitrary positive integer. It suffices to show that there is a $G$-module $V$ such that $\beta\left(K[V]^G\right) > k$. As observed above, there is a faithful $G$-module $U$. By Proposition 2.1 there is an irreducible $G$-module $X$ which does not occur as a composition factor of $K[U]_0$, $K[U]_1$, ..., $K[U]_{k-1}$, but which occurs as a composition factor of $K[U]_m$ for some $m \geqslant k$. We may assume that $m$ is minimal with this property.

Let $W$ be a submodule of $K[U]_m$ of smallest possible dimension such that $X$ is a composition factor of $W$. Thus, if $W'$ is any proper submodule of $W$, $X$ is not a composition factor of $W'$, so $X$ is a composition factor of $W/W'$. Take $\varphi \in \mathrm{Hom}_G(W, K[U]_j)$ with $j < m$. Since $X$ is not a composition factor of $K[U]_j$, the above argument shows that $\ker(\varphi)$ cannot be a proper submodule of $W$. Hence $\varphi = 0$. Therefore $\mathrm{Hom}_G(W, K[U]_j) = 0$ for all $j < m$. But, clearly, $\mathrm{Hom}_G(W, K[U]_m) \neq 0$. Set $V = W \oplus U$.

We proceed essentially as in the proof of Derksen and Kemper [2, Proposition 1.2]. The algebra $K[V]$ can be identified with $K[W] \otimes_K K[U]$ and thus has a $G$-invariant bigrading, $K[V] = \bigoplus K[V]_{i,j}$, where $K[V]_{i,j} = K[W]_i \otimes_K K[U]_j$. For all $l \in \mathbb{N}$, $K[V]_l = \bigoplus_{i+j=l} K[V]_{i,j}$ and $K[V]_l^G = \bigoplus_{i+j=l} K[V]_{i,j}^G$. Furthermore, for all $j$,

$$K[V]_{1,j}^G = (W^* \otimes_K K[U]_j)^G \cong \mathrm{Hom}_G(W, K[U]_j).$$

Hence $K[V]_{1,j}^G = 0$ for $j < m$ but $K[V]_{1,m}^G \neq 0$. Let $A$ be the direct sum of all $K[V]_{i,j}^G$ with $i \neq 1$. Note that $A$ is a subalgebra of $K[V]^G$. Since $K[V]_{1,j}^G = 0$ for $j < m$ it follows that $K[V]_0^G, \ldots, K[V]_m^G$ are contained in $A$. But $K[V]_{m+1}^G$ is not contained in $A$ because $K[V]_{1,m}^G \neq 0$. Therefore $\beta\left(K[V]^G\right) \geqslant m + 1 > k$, as required. $\square$

We shall now consider the general case. If $N$ is a closed normal subgroup of $G$ we have

$$\beta(G/N) \leqslant \beta(G), \tag{2.1}$$

since every $G/N$-module is also a $G$-module. Moreover, if $H$ is a closed subgroup of $G$ of finite index, then by Schmid [12, Proposition 5.1] we have

$$\beta(H) \leqslant \beta(G). \tag{2.2}$$

(Schmid stated this result for finite groups, but the proof only uses that the index is finite.)

Suppose that $G$ is infinite. We wish to prove that $\beta(G) = \infty$. By (2.2) we may assume that $G$ is connected. By Proposition 2.2 we may also assume that $G$ is unipotent. Thus, by Humphreys [8, Theorem 19.3], $G$ has a closed normal subgroup $N$ such that $\dim(G/N) = 1$. Also $G/N$ is unipotent by [8, Theorem 15.3(c)]. Thus, by [8, Theorem 20.5], $G/N$ is isomorphic to the additive group of $K$. We identify this group with the group of upper unitriangular matrices $\mathrm{U}_2(K)$. By (2.1) it suffices to prove that $\beta\left(\mathrm{U}_2(K)\right) = \infty$.

## 3    Isomorphisms of spaces of invariants

The results proved in this section are closely connected with results described by Grosshans [7] in relation to the "transfer principle". Such results have a long history, as outlined in [7, Chapter 2, Introduction]. In particular, Roberts [11] in 1861 (the year is often given as 1871) introduced an isomorphism, often now called Roberts' isomorphism, which is the special case $G = \mathrm{SL}_2(K)$ of the isomorphism given by Corollary 3.5 below (see Example 3.6), and which in characteristic 0 has Weitzenböck's theorem (see Grosshans [7, Theorem 10.1]) as a consequence. Other references include Seshadri [13], Fauntleroy [4], and Tyc [14]. All these are concerned with the special case mentioned above. Our treatment has the merits that it is elementary, self-contained and formulated rather generally. (Only Theorem 3.4 needs anything non-trivial from the theory of algebraic groups, and this result can be bypassed in our application by the use of Theorem 3.2.)

Let $G$ be any group. Furthermore, let $K$ be any field and $V$ a $K$-space. We write $\mathcal{F}(G, V)$ for the $K$-space consisting of all functions from $G$ to $V$: it is a $K$-algebra if $V$ is a $K$-algebra, as in the case $V = K$. Let $F$ be a subspace of $\mathcal{F}(G, K)$ and consider the tensor product $F \otimes V$ (all tensor products will be taken over $K$). Each element $\alpha = \sum f_i \otimes v_i$ of $F \otimes V$ determines an element $\overline{\alpha}$ of $\mathcal{F}(G, V)$ satisfying $\overline{\alpha}(g) = \sum f_i(g)v_i$ for all $g \in G$. By choosing the $v_i$ to be linearly independent it is easily seen that $\alpha \mapsto \overline{\alpha}$ is an embedding. Thus we often identify $\alpha$ with $\overline{\alpha}$ and regard $F \otimes V$ as a subspace of $\mathcal{F}(G, V)$.

For a subgroup $H$ of $G$ we define

$$F_{G/H} = \{f \in F : f(gh) = f(g) \quad \text{for all} \quad g \in G, \ h \in H\}.$$

This is the subspace of $F$ consisting of all functions which are constant on the left cosets $gH$ of $H$. (It can also be thought of as consisting of the functions which are invariant under the action of $H$ by right translation.) Then, regarding $F \otimes V$ and $F_{G/H} \otimes V$ as subspaces of $\mathcal{F}(G, V)$, we easily see that

$$F_{G/H} \otimes V = \{\alpha \in F \otimes V : \alpha(gh) = \alpha(g) \text{ for all } g \in G, \ h \in H\}. \tag{3.1}$$

Suppose that $V$ is a $KG$-module (not necessarily finite-dimensional). Then we define

$$(F \otimes V)^G = \{\alpha \in F \otimes V : \alpha(gg') = g\alpha(g') \text{ for all } g, g' \in G\} \tag{3.2}$$

and

$$(F_{G/H} \otimes V)^G = (F \otimes V)^G \cap (F_{G/H} \otimes V). \qquad (3.3)$$

These may be interpreted as spaces of $G$-invariants when $F$ is closed under the action of $G$ on $\mathcal{F}(G, K)$ by left translations because in that case $F \otimes V$ may be regarded as a $KG$-module under the diagonal action of $G$, with $F_{G/H} \otimes V$ as a submodule, and $(F \otimes V)^G$ and $(F_{G/H} \otimes V)^G$ are the spaces of $G$-invariants in the usual sense.

We say that *V has coefficients in F* if there is a $K$-linear function $\rho : V \to F \otimes V$ such that the action of $G$ is given by

$$gv = (\rho(v))(g) \quad \text{for all} \quad g \in G, \ v \in V. \qquad (3.4)$$

Equivalently, with respect to any basis $\{v_i : i \in I\}$ of $V$, the functions $f_{i,j}$ satisfying $gv_i = \sum_j f_{i,j}(g)v_j$ all belong to $F$, and for each $i$ there are only finitely many $j \in I$ with $f_{i,j} \neq 0$. (The connection is given by $\rho(v_i) = \sum_j f_{i,j} \otimes v_j$ for all $i$.) In this terminology, if $G$ is an algebraic group, then a finite-dimensional $KG$-module is a $G$-module if and only if it has coefficients in $K[G]$. Moreover, if a linear algebraic group $G$ acts on an affine variety $W$ by a morphism $G \times W \to W$, then $K[W]$, the ring of regular functions on $W$, clearly has coefficients in $K[G]$.

We can now formulate a generalisation of the "transfer principle" for algebraic groups as stated in Grosshans [7, Theorem 9.1]. The proof is elementary and similar to the proof in [7].

**Theorem 3.1.** *Let $G$ be any group, $H$ a subgroup, and $K$ a field. Let $F$ be any space of functions from $G$ to $K$ and let $V$ be a $KG$-module with coefficients in $F$. Then there is an isomorphism of $K$-spaces*

$$\Phi : (F_{G/H} \otimes V)^G \longrightarrow V^H$$

*given by $\Phi(\alpha) = \alpha(e)$ for all $\alpha \in (F_{G/H} \otimes V)^G$, where $e$ denotes the identity element of $G$.*

*Proof.* For $\alpha \in (F_{G/H} \otimes V)^G$ and $h \in H$, (3.2) and (3.1) give

$$h\alpha(e) = \alpha(h) = \alpha(e).$$

Thus $\alpha(e) \in V^H$. Therefore $\Phi$ is well-defined. Clearly $\Phi$ is $K$-linear. If $\Phi(\alpha) = 0$ then $0 = g\alpha(e) = \alpha(g)$ for all $g \in G$, by (3.2). Thus $\alpha = 0$. Therefore $\Phi$ is injective.

To prove surjectivity take $v \in V^H$ and set $\alpha = \rho(v) \in F \otimes V$, with $\rho$ as in (3.4). Thus $gv = \alpha(g)$ for all $g \in G$. Hence, for all $g \in G, h \in H$,

$$\alpha(gh) = (gh)v = g(hv) = gv = \alpha(g).$$

Thus, by (3.1), $\alpha \in F_{G/H} \otimes V$. Also, for all $g, g' \in G$,

$$\alpha(gg') = (gg')v = g(g'v) = g\alpha(g').$$

Thus, by (3.2), $\alpha \in (F \otimes V)^G$. Hence, by (3.3), $\alpha \in (F_{G/H} \otimes V)^G$. Finally,

$$\Phi(\alpha) = \alpha(e) = ev = v.$$

Therefore $\Phi$ is surjective. $\qquad \square$

The problem with Theorem 3.1 is that it is usually not easy to work with $F_{G/H}$, for example in the case that $F = K[G]$ for $G$ an algebraic group. The purpose of the next two results is to derive isomorphisms between $F_{G/H}$ and objects which are better to work with.

Suppose now that $K$ is any *infinite* field. If $V$ and $W$ are finite-dimensional $K$-spaces and $X \subseteq V$, a function $\varphi : X \to W$ is called a *polynomial function* if there are $\dim W$ polynomials over $K$ in $\dim V$ variables which give the coordinates of $\varphi(v)$ in terms of the coordinates of $v$ for all $v \in X$ (with respect to fixed but arbitrary bases of $V$ and $W$). Furthermore $K[X]$ denotes the algebra of all polynomial functions from $X$ to $K$. Recall that a linear algebraic group $G$ is by definition a Zariski-closed subset of some vector space $K^m$, and in this context $K[G]$ is just the ring of regular functions on $G$.

Let $U$ be a finite-dimensional $K$-space and let $K(U)$ denote the field of quotients of $K[U]$. Let $\chi = \xi/\xi' \in K(U)$ where $\xi, \xi' \in K[U]$ and $\xi' \neq 0$. For $u \in U$ such that $\xi'(u) \neq 0$ we define $\chi(u) = \xi(u)/\xi'(u) \in K$. For $d \in K[U] \setminus \{0\}$ let $U_d = \{u \in U : d(u) \neq 0\}$ and let $K[U]_d$ denote the set of elements of $K(U)$ of the form $\xi/d^r$ where $\xi \in K[U]$ and $r \in \mathbb{N}$.

**Theorem 3.2.** *Let $K$ be an infinite field and $G$ a group such that $G \subseteq K^m$, for some $m \in \mathbb{N}$. Let $U$ be a finite-dimensional $KG$-module with coefficients in $K[G]$. Let $x \in U$ and let $\pi : G \to U$ be defined by $\pi(g) = gx$ for all $g \in G$. Write $G_x = \{g \in G : gx = x\}$. Let $d_1, \ldots, d_n$ be non-zero elements of $K[U]$ and, for $i = 1, \ldots, n$, let $\varphi_i : U_{d_i} \to G$ be a function given by elements $\xi_i^{(1)}, \ldots, \xi_i^{(m)}$ of $K[U]_{d_i}$ such that $\varphi_i(u) = \left( \xi_i^{(1)}(u), \ldots, \xi_i^{(m)}(u) \right) \in G$ for all $u \in U_{d_i}$. Suppose that*

(a) *$\pi \circ \varphi_i$ is the identity on $U_{d_i}$, for $i = 1, \ldots, n$,*

(b) *$K[U]_{d_1} \cap \cdots \cap K[U]_{d_n} = K[U]$ (that is, $d_1, \ldots, d_n$ are coprime), and*

(c) *$\pi(G) \subseteq U_{d_1} \cup \cdots \cup U_{d_n}$.*

*Then there is an algebra isomorphism $\pi^* : K[U] \to K[G]_{G/G_x}$ given by $\pi^*(\xi) = \xi \circ \pi$ for all $\xi \in K[U]$.*

*Proof.* Let $\pi^* : K[U] \to \mathcal{F}(G, K)$ be defined by $\pi^*(\xi) = \xi \circ \pi$ for all $\xi \in K[U]$. It is easily verified that $\pi^*$ is an algebra homomorphism. Since $U$ has coefficients in $K[G]$, the function $\pi : G \to U$ is a polynomial function. Hence $\xi \circ \pi \in K[G]$ for all $\xi \in K[U]$. Since $\pi$ is constant on the left cosets of $G_x$, so is $\xi \circ \pi$. Thus $\pi^*(K[U]) \subseteq K[G]_{G/G_x}$. Hence we have $\pi^* : K[U] \to K[G]_{G/G_x}$.

Let $\xi \in K[U]$ satisfy $\pi^*(\xi) = 0$. Then, for all $u \in U_{d_1}$, we have $(\xi \circ \pi)(\varphi_1(u)) = 0$. But, by (a), $\pi \circ \varphi_1$ is the identity on $U_{d_1}$. Hence $\xi(u) = 0$ for all $u \in U_{d_1}$. Thus $\xi(u)d_1(u) = 0$ for all $u \in U$, and so $\xi = 0$. Therefore $\pi^*$ is injective.

To prove surjectivity, let $f \in K[G]_{G/G_x} \subseteq K[G]$ and think of $f$ as a polynomial in $m$ variables. With the $\xi_i^{(j)}$ as in the statement of the theorem, define $\chi_i = f(\xi_i^{(1)}, \ldots, \xi_i^{(m)})$. Thus $\chi_i \in K[U]_{d_i}$ and

$$\chi_i(u) = (f \circ \varphi_i)(u) \quad \text{for all} \quad u \in U_{d_i}. \tag{3.5}$$

Let $g \in \pi^{-1}(U_{d_i})$. Then, by (a), $\pi(\varphi_i(\pi(g))) = \pi(g)$ and so $\varphi_i(\pi(g))G_x = gG_x$. Since $f \in K[G]_{G/G_x}$ we obtain $f(\varphi_i(\pi(g))) = f(g)$. Therefore, by (3.5),

$$\chi_i(\pi(g)) = f(g) \quad \text{for all} \quad g \in \pi^{-1}(U_{d_i}). \tag{3.6}$$

Let $i, j \in \{1, \ldots, n\}$ and $u \in U_{d_i} \cap U_{d_j}$. Write $g = \varphi_i(u)$. Then, by (a), $\pi(g) = u \in U_{d_i} \cap U_{d_j}$. Hence, by (3.6) applied to both $i$ and $j$,

$$\chi_i(u) = f(g) = \chi_j(u). \tag{3.7}$$

Write $\chi_i - \chi_j = \eta / \left( d_i^{r_i} d_j^{r_j} \right)$ where $\eta \in K[U]$. Then, by (3.7), $\eta(u) d_i(u) d_j(u) = 0$ for all $u \in U$. Thus $\eta = 0$ and so $\chi_i = \chi_j$. Write $\chi$ for the element such that $\chi = \chi_i$ for all $i$. Then, by (b), $\chi \in K[U]$.

Let $g$ be any element of $G$. Then, by (c), there exists $i$ such that $g \in \pi^{-1}(U_{d_i})$. Hence, by (3.6),

$$f(g) = \chi(\pi(g)) = (\chi \circ \pi)(g) = (\pi^*(\chi))(g).$$

Therefore $f = \pi^*(\chi)$ and so $\pi^*$ is surjective. $\qquad \square$

The following example illustrates how the hypotheses of Theorem 3.2 can be easily verified.

*Example 3.3.* Consider the group $G = \mathrm{SL}_2(K) \subset K^4$ ($K$ any infinite field), and let $U = K^2$ be the natural 2-dimensional $G$-module. Consider the point $x = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in U$. Clearly $G_x = \mathrm{U}_2(K)$ is the additive group. The map $\pi : G \to U$ is given by

$$\pi \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix}.$$

Define $d_1, d_2 \in K[U]$ and $\varphi_1 : U_{d_1} \to G$, $\varphi_2 : U_{d_2} \to G$, by

$$d_1 \begin{pmatrix} s \\ t \end{pmatrix} = s, \qquad d_2 \begin{pmatrix} s \\ t \end{pmatrix} = t,$$

$$\varphi_1 \begin{pmatrix} s \\ t \end{pmatrix} = \begin{pmatrix} s & 0 \\ t & s^{-1} \end{pmatrix}, \quad \varphi_2 \begin{pmatrix} s \\ t \end{pmatrix} = \begin{pmatrix} s & -t^{-1} \\ t & 0 \end{pmatrix}.$$

Then it is easy to verify that the hypotheses of Theorem 3.2 hold.

Example 3.3 extends in a straightforward way to the case $G = \mathrm{SL}_n(K)$ for $n \geqslant 2$. In that case we obtain a point stabilizer $G_x$ which is isomorphic to the semidirect product $K^{n-1} \rtimes \mathrm{SL}_{n-1}(K)$.

Theorem 3.2 is sufficient for our application. However, for purposes of comparison, we give a parallel theorem for algebraic groups. It is less elementary than Theorem 3.2 but can be proved quite quickly from standard results. The theorem is essentially well known. It has some overlap with Grosshans [7, Theorem 4.3 and Theorem 9.3], and the terminology follows Borel [1].

**Theorem 3.4.** *Let $G$ be a linear algebraic group over an algebraically closed field $K$ and suppose that $G$ acts on a normal, irreducible, affine variety $U$ by a morphism $G \times U \to U$. Let $x$ be an element of $U$ such that*

*(a) $Gx$ (the $G$-orbit of $x$) is open in $U$,*

*(b) $\dim(U \setminus Gx) \leqslant \dim U - 2$, and*

*(c) the morphism $\pi : G \to Gx$ given by $g \mapsto gx$ is separable.*

*Then there is an algebra isomorphism* $\pi^* : K[U] \to K[G]_{G/G_x}$ *given by* $\pi^*(\xi) = \xi \circ \pi$ *for all* $\xi \in K[U]$.

*Proof.* We may view $\pi$ as a morphism $\pi : G \to U$. For $\xi \in K[U]$ we have $\xi \circ \pi \in K[G]$. Since $\pi$ is constant on the left cosets of $G_x$, so is $\xi \circ \pi$. Hence $\pi^* : K[U] \to K[G]_{G/G_x}$ may be defined by $\pi^*(\xi) = \xi \circ \pi$ for all $\xi$. Clearly $\pi^*$ is an algebra homomorphism. If $\pi^*(\xi) = 0$ then $\xi$ vanishes on $\pi(G) = Gx$ and so $\xi = 0$ by (a). Hence $\pi^*$ is injective.

To prove surjectivity take $f \in K[G]_{G/G_x}$. Thus $f$ is constant on the left cosets of $G_x$. By (c) and Borel [1, Proposition 6.7], $\pi : G \to Gx$ is a quotient morphism. Hence, by [1, §6.1], $\pi$ has the universal mapping property. Therefore there exists a morphism $\widetilde{\chi} : Gx \to K$ such that $\widetilde{\chi} \circ \pi = f$. By Grosshans [7, Theorem 4.2] or Eisenbud [3, Corollary 11.4 and remark following], (a) and (b) imply that $\widetilde{\chi}$ extends to $\chi \in K[U]$. Thus $\pi^*(\chi) = \chi \circ \pi = \widetilde{\chi} \circ \pi = f$. Therefore $\pi^*$ is surjective.                         $\square$

If $G$ and $U$ are as in Theorem 3.2 or Theorem 3.4, $G$ acts on $K[U]$ by left translations. Thus, if $V$ is a $KG$-module, $G$ acts diagonally on $K[U] \otimes V$ and we may consider the space of invariants $(K[U] \otimes V)^G$. For the same reasons as given for $F \otimes V$ at the beginning of this section, we may regard the elements of $K[U] \otimes V$ as functions from $U$ to $V$. The following corollary is particularly interesting in the case where $G$ is a linear algebraic group acting on an affine variety $W$ (or, in particular, a finite-dimensional $K$-space) by a morphism $G \times W \to W$, and $V = K[W]$. With $x \in U$ as before, the corollary relates $G_x$-invariant regular functions on $W$ to $G$-invariant regular functions on the larger variety $U \times W$.

**Corollary 3.5.** *Suppose that the hypotheses of Theorem 3.2 or 3.4 are satisfied. Let $V$ be a $KG$-module with coefficients in $K[G]$. Then there is an isomorphism of $K$-spaces*

$$\varphi : (K[U] \otimes V)^G \longrightarrow V^{G_x}$$

*given by* $\varphi(\alpha) = \alpha(x)$ *for all* $\alpha \in (K[U] \otimes V)^G$.

*Proof.* By Theorem 3.2 or Theorem 3.4 we have an isomorphism

$$\pi^* \otimes \mathrm{id}_V : K[U] \otimes V \longrightarrow K[G]_{G/G_x} \otimes V.$$

It is easily verified that $\pi^*$ is $G$-equivariant with respect to the actions of $G$ by left translation. Thus $\pi^* \otimes \mathrm{id}_V$ restricts to an isomorphism

$$\pi' : (K[U] \otimes V)^G \longrightarrow (K[G]_{G/G_x} \otimes V)^G.$$

Let $\Phi$ be the isomorphism of Theorem 3.1 with $F = K[G]$ and $H = G_x$, and define $\varphi = \Phi \circ \pi'$. Thus $\varphi : (K[U] \otimes V)^G \to V^{G_x}$ is an isomorphism. For $\alpha \in (K[U] \otimes V)^G$ with $\alpha = \sum \xi_i \otimes v_i$ we have

$$\varphi(\alpha) = \Phi\Big(\sum (\xi_i \circ \pi) \otimes v_i\Big) = \sum (\xi_i \circ \pi)(e)v_i = \sum \xi_i(x)v_i = \alpha(x).$$

$\square$

*Example 3.6.* This is a continuation of Example 3.3. Let $G = \mathrm{SL}_2(K)$ with $K$ any infinite field, and let $U = K^2$ be the natural $G$-module. Then, for any $KG$-module $V$ with coefficients in $K[G]$, Corollary 3.5 yields

$$(K[U] \otimes V)^{\mathrm{SL}_2(K)} \cong V^{\mathrm{U}_2(K)}.$$

In the special case where $V = K[W]$ with $W$ a $G$-module, we obtain

$$K[U \oplus W]^{\mathrm{SL}_2(K)} \cong K[W]^{\mathrm{U}_2(K)}.$$

This is known as Roberts' isomorphism. It is usually only formulated and proved for the case where $K$ is algebraically closed.

For our purposes, it is interesting to draw the following two consequences. Note that $U^G$ is always non-empty if $U$ is a $KG$-module.

**Corollary 3.7.** *Suppose that the hypotheses of Theorem 3.2 or 3.4 are satisfied. Suppose further that $U^G$ is non-empty. Let $V$ be a $KG$-module with coefficients in $K[G]$. Then there is a homomorphism of $K$-spaces*

$$\kappa : V^{G_x} \longrightarrow V^G$$

*which restricts to the identity on $V^G$. If $V$ has the structure of a graded vector space or a $K$-algebra which is respected by the $G$-action, then this structure is also respected by $\kappa$.*

*Proof.* Let $y \in U^G$. Let $\varphi : (K[U] \otimes V)^G \to V^{G_x}$ be the isomorphism of Corollary 3.5. For $\alpha \in (K[U] \otimes V)^G$ it is easy to check that $\alpha(y) \in V^G$. Thus we may define $\vartheta : (K[U] \otimes V)^G \to V^G$ by $\vartheta(\alpha) = \alpha(y)$ for all $\alpha$. Set $\kappa = \vartheta \circ \varphi^{-1}$. Thus $\kappa : V^{G_x} \to V^G$.

Let 1 be the identity element of $K[U]$, that is, the constant function 1. Then, for all $v \in V^G$, we have $1 \otimes v \in (K[U] \otimes V)^G$ and $\varphi(1 \otimes v) = v$. Thus

$$\kappa(v) = (\vartheta \circ \varphi^{-1})(\varphi(1 \otimes v)) = \vartheta(1 \otimes v) = v.$$

Hence $\kappa$ restricts to the identity map on $V^G$.

Let $V$ be graded, $V = \bigoplus_{i \in I} V_i$, with $I$ any index set and $GV_i \subseteq V_i$ for all $i$. Then $K[U] \otimes V$ acquires a grading with $(K[U] \otimes V)_i = K[U] \otimes V_i$. Intersections with spaces of invariants yield gradings of $V^G$, $V^{G_x}$ and $(K[U] \otimes V)^G$. Clearly $\varphi$ and $\vartheta$ preserve the gradings. Since $\varphi$ is an isomorphism, $\varphi^{-1}$ also preserves the gradings; hence so does $\kappa$.

Now assume that $V$ is a $K$-algebra and $G$ acts by algebra automorphisms. Then $K[U] \otimes V$ is a $K$-algebra in the obvious way and so are $V^G$, $V^{G_x}$, and $(K[U] \otimes V)^G$. It is easy to see that $\varphi$ and $\vartheta$ are algebra homomorphisms; hence so is $\kappa$. $\qquad\square$

**Corollary 3.8.** *Let $G$ be a linear algebraic group and suppose that the hypotheses of Theorem 3.2 or 3.4 are satisfied. Suppose further that $U^G$ is non-empty. Then*

$$\beta(G_x) \geqslant \beta(G).$$

*Proof.* Let $V$ be any $G$-module. Then $K[V]$ is a $KG$-module with coefficients in $K[G]$. Corollary 3.7 yields a degree-preserving epimorphism $K[V]^{G_x} \to K[V]^G$ of $K$-algebras. It follows that $\beta\left(K[V]^{G_x}\right) \geqslant \beta\left(K[V]^G\right)$. Therefore $\beta(G_x) \geqslant \beta(G)$, as required. $\qquad\square$

To complete the proof of Theorem 1.1 we apply Corollary 3.8. In Example 3.3 we have already verified the hypotheses of Theorem 3.2 for $G = \mathrm{SL}_2(K)$ and $U$ the natural $G$-module (with $G_x = \mathrm{U}_2(K)$). (Alternatively, Theorem 3.4 may be used.) Also, $U^G$ is non-empty because $0 \in U^G$. Therefore, by Corollary 3.8, $\beta\left(\mathrm{U}_2(K)\right) \geqslant \beta\left(\mathrm{SL}_2(K)\right)$. But $\beta\left(\mathrm{SL}_2(K)\right) = \infty$ by Proposition 2.2. This completes the proof.

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Roger M. Bryant                     Gregor Kemper
Department of Mathematics           Technische Universität München
UMIST                               Zentrum Mathematik - M11
PO Box 88                           Boltzmannstr. 3
Manchester M60 1QD                  85 748 Garching
United Kingdom                      Germany
bryant@umist.ac.uk                  kemper@ma.tum.de