ON THE BRANCH LOCUS OF QUOTIENTS BY FINITE GROUPS AND THE DEPTH OF THE ALGEBRA OF INVARIANTS

NIKOLAI GORDEEV AND GREGOR KEMPER

ABSTRACT. Let $A = B^G$, where B is a Noetherian algebra over a field K of characteristic $p \neq 0$ and G is a finite group such that p divides $|G|$. We give estimates for the depth of A in terms of the codimension of the branch locus of the extension B/A.

1. Introduction

Let B be a commutative Noetherian ring and G a finite group of automorphisms. Further, let $A = B^G$ be the invariant ring. One of the main goals of invariant theory is the comparison of properties of A and B in terms of the properties of the extension. In the classical case the ring $B = S(V)$ is a symmetric algebra of a linear space V, that is, a polynomial algebra, and $G \leq GL(V)$. In this case the ring B is "good" and the question is what kind of "goodness" we can be expect of A under what conditions on G and the extension B/A. A typical example is the Chevalley-Shephard-Todd Theorem which settles the question of when $S(V)^G$ is a polynomial algebra (under the assumption that the ground field has characteristic zero): "if and only if G is generated by pseudo-reflections". This statement can be reformulated in terms of the "branch locus" of the extension (see [N]). Namely, the algebra $S(V)^G$ is a polynomial algebra "if and only if the branch locus of the extension $S(V)/S(V)^G$ is *pure* and has codimension one" (note that "only if" follows directly from the Zariski-Nagata Purity Theorem [N]). There are more examples of the influence of the branch locus, that is, the set of *ramified* prime ideals of $S(V)$, on the structure of $S(V)^G$. For instance the Grothendieck Purity Theorem for complete intersections [Gr] shows that if $S(V)^G$ is a complete intersection then the branch locus of $S(V)$ has codimension ≤ 2, and if it has codimension 2 it is *pure*. (This fact was reformulated in [KW], [G1] in the following way: If the algebra $S(V)^G$ is a complete intersection then the group G is generated by elements g such that $\operatorname{rank}(g-1) \leq 2$.) In [G2] it was shown that the homological dimension of $S(V)^G$ can be estimated by the maximal codimension of an

This work was done during a visit of the first author to the Forschergruppe "Arithmetik" at the Universities Heidelberg and Mannheim. The first author received partial support from the grant RFFI-00-01-00441.

irreducible component of the branch locus (or in linear group language by min rank $(g-1),\ g \in G,\ g \neq 1$.)

Recently, the second author [K1, 2] gave formulas which allow to estimate (and sometimes to calculate) the depth of the algebra $S(V)^G$ in the case when the ground field K has characteristic which divides $|G|$. (If char K does not divide $|G|$ then depth $S(V)^G = \dim V$.) These estimates were obtained by using heights of annihilators of non-zero elements of cohomology groups $H^q(G, S(V))$ and the number $r = \min\{q > 0 \mid H^q(G, S(V)) \neq 0\}$. In [LP] some of these result were extended to more general extensions B/A. In particular, these estimates imply that if $S(V)^G$ is a Cohen-Macaulay algebra and G is a p-group where $p = $ char K, then $G = \langle g \mid \operatorname{rank}(g-1) \leq 2\rangle$. (This is reminiscent of the situation of complete intersections as invariant algebras.) This result can also be reformulated in the language of branch loci. Thus, at least in the case of p-groups, one can see again the dependence of properties of invariant algebra on properties of the branch locus. (The results [K1, 2] show that in the "mixed case": $p \mid |G|$, $|G| \neq p^k$, the situation is much more complicated.)

This paper is devoted to a generalization of the results [K1, 2], with emphasis on connections with the branch locus. Namely, we consider the case where B is a finitely generated Cohen-Macaulay domain over a field K of characteristic p. Suppose that the wild ramification locus X_{wr} in $X = \operatorname{Spec} B$ is non-empty. Then we obtain:

$$\operatorname{depth} A \leq \dim X_{wr} + 2|G_p|,$$

where $G_p \leq G$ is a Sylow p-subgroup (see Corollary 5.12). If we are in the standard situation $B = S(V)$, then with $k := \dim A - \operatorname{depth} A + 2$ we obtain that G is generated by elements of order not divisible by char(K) and by k-reflections (as defined at the beginning of Section 5.1). In fact, we get slightly more technical results for a more general situation (see Theorems 5.5 and 5.9).

We obtain stronger results in the case where B is a normal Cohen-Macaulay algebra and G is a p-group. In fact,

$$\operatorname{depth} A \ \leq\ \dim A - c_s + 2,$$

where c_s is maximum of codimensions of all irreducible components of all branch loci corresponding to extensions B'/A for normal subalgebras $A \subset B' \subset B$ (see Theorem 6.1). If again we set $k := \dim A - \operatorname{depth} A + 2$ in this case, we obtain that every inertia subgroup of G is generated by k-reflections.

Acknowledgment. We thank the anonymous referee for helpful comments on the paper. In particular, the inequality (1) was brought to our attention by the referee.

2. Notation and terminology

2.1. Let A be a commutative ring and M an A-module. Further, let $1 \in U \subset A$ be a multiplicative subset. Then we write A_U for the corresponding localization and M_U for the A_U-module $M \otimes_A A_U$. If $\mathfrak{p}$ is a prime ideal of A, the symbol $A_{\mathfrak{p}}$ means $A_{A \setminus \mathfrak{p}}$.

If $\mathfrak{a}$ is an ideal of A, the symbol $V(\mathfrak{a})$ stands for the closed subset in Spec A of all prime ideals containing $\mathfrak{a}$.

If A is a local ring, the symbols $\widehat{A}, \widehat{M}$ mean the completion of the ring A and the module M with respect to the $\mathfrak{m}$-adic topology ($\mathfrak{m}$ being the maximal ideal of A).

The symbol $\mathrm{ht}(\mathfrak{a})$ stands for the height of the ideal $\mathfrak{a}$.

All rings and modules below are supposed to be Noetherian.

A ring is called equidimensional if all its maximal ideals have the same height. For example, if a finitely generated algebra over a field has no zero-divisors, then it is equidimensional (see [Eis, Corollary 13.4]).

An equidimensional ring is called *geometric* if it is a localization (with respect to any multiplicative set U) of a finitely generated K-algebra without zero-divisors, where K is a field.

The symbol F_p denotes the prime field of characteristic p.

2.2. Let $A \subset B$ be commutative rings. The ring B is called a *separable* A-algebra if B is projective as a $B \otimes_A B$-module (where the action is given by $(b_1 \otimes b_1)(b) = b_1 b b_2$) (see [AB]). If $\phi : B \otimes_A B \longrightarrow B$ is the homomorphism defined by $\phi(b_1 \otimes b_2) = b_1 b_2$ and $\mathfrak{I} = \mathrm{Ann}\ \ker\ \phi$ then the ideal $\phi(\mathfrak{I})$ is called the *homological* or *Noetherian different* of the extension B/A and denoted by $\mathfrak{N}_{B/A}$. The A-algebra B is separable if and only if $\mathfrak{N}_{B/A} = B$ (see [AB]).

Now let $B^G = A$, where G is a group of automorphisms of the ring B. Let $\mathfrak{p} \in \mathrm{Spec}\ B$. The stabilizer of $\mathfrak{p}$ in G is written as $G_{\mathfrak{p}}$ and the inertia subgroup as $I_{\mathfrak{p}} := \{g \in G \mid (g-1)B \subseteq \mathfrak{p}\}$.

2.3. Now suppose an A-module M is also an $A[G]$-module for a finite group G. Then $tr : M \longrightarrow M^G$ denotes the trace map. If $H \leq G$ then $tr_{G/H} : M^H \longrightarrow M^G$ denotes the relative trace map.

2.4. Let M be an $A[G]$-module. We will also consider M as a $\mathbb{Z}[G]$-module. The corresponding cohomology groups of M as $\mathbb{Z}[G]$-module are written as $H^q(G, M)$, $q \geq 0$, and the homology groups as $H_q(G, M)$, $q \geq 0$. Also, we use Tate cohomology groups (see [AW]): $\widehat{H}^q(G, M)$, $q \in \mathbb{Z}$ which coincide with $H^q(G, M)$ if $q > 0$ and with $H_{-q-1}(G, M)$ if $q < -2$. For $q = 0, -1$ we have

$$\widehat{H}^0(G, M) = M^G / tr(M), \quad \widehat{H}^{-1} = \ker(tr : M/I_G M \longrightarrow M^G)$$

(where I_G is the kernel of the map $\mathbb{Z}[G] \longrightarrow \mathbb{Z}$ sending each $g \in G$ to 1).

3. Some known properties of cohomology groups

3.1. As in 2.3, let A be a commutative ring and M an $A[G]$-module, where G is a finite group. We may consider the abelian groups $\widehat{H}^q(G, M)$ as finitely generated A-modules (indeed, these cohomology groups can be defined as cohomology of a complex of $\mathbb{Z}[G]$-modules which are also finitely generated A-modules and, moreover, all differentials of a this complex are homomorphisms of A-modules [AW]). Let A' be a flat A-algebra. Since all cohomology groups appear in a complex of A-modules we have:

$$\widehat{H}^q(G, M) \otimes_A A' = \widehat{H}^q(G, M \otimes_A A').$$

In particular, if $U \subset A$ is a multiplicative set we have:

$$\widehat{H}^q(G, M)_U = \widehat{H}^q(G, M_U)$$

for every $q \in \mathbb{Z}$. In particular, this implies that $\widehat{H}^q(G, M_U) = 0$ if and only if for every $c \in \widehat{H}^q(G, M)$ there exists and element $l \in U$ such that $lc = 0$.

3.2. For every normal subgroup H of G we have an exact sequence

$$0 \longrightarrow \widehat{H}^1(G/H, M^H) \longrightarrow \widehat{H}^1(G, M) \longrightarrow \widehat{H}^1(H, M)$$

[AW, Proposition 5.1]. More generally, if $\widehat{H}^i(H, M) = 0$ for every $1 \leq i \leq q-1$ then we have an exact sequence

$$0 \longrightarrow \widehat{H}^q(G/H, M^H) \longrightarrow \widehat{H}^q(G, M) \longrightarrow \widehat{H}^q(H, M)$$

[AW, Proposition 5.2]. In this case we will identify the group $\widehat{H}^q(G/H, M^H)$ with a subgroup of $\widehat{H}^q(G, M)$.

3.3. Let p be a prime such that $pM = 0$. Assume that G is a p-group. Then the following statements are equivalent:

(i) M is a free $\mathbb{F}_p[G]$-module;
(ii) $\widehat{H}^q(G, M) = 0$ for every q;
(iii) $\widehat{H}^q(G, M) = 0$ for some q

[AW, Proposition 9.1]

3.4. Let p be a prime number and let $|G| = p^a m$ where $(p,\, m) = 1$. Further, let $H \leq G$ be a cyclic group of the order p. Then there exists a positive integer $r \leq 2p^{a-1}(p-1)$ and an $\alpha \in \widehat{H}^r(G, F_p)$ such that $\mathrm{res}_H(\alpha) \in \widehat{H}^r(H, F_p)$ is non-zero (see the proof of Theorem 4.1.3 in [Be]).

4. The branch locus and the total branch locus

4.1. **The branch locus for a finite extension of commutative rings.** Let $A \subset B$ be commutative Noetherian rings such that B is finitely generated as an A-module. Let $X = \text{Spec } B$, $Y = \text{Spec } A$.

Let $\mathfrak{p} \in \text{Spec } B$ and $\mathfrak{p}' = \mathfrak{p} \cap A$. We say that $\mathfrak{p}$ is *unramified* in X/Y (or in B/A) if $B_\mathfrak{p}/\mathfrak{p}'B_\mathfrak{p}$ is a field which is a separable extension of the field $A_{\mathfrak{p}'}/\mathfrak{p}'A_{\mathfrak{p}'}$ [AB]. Otherwise $\mathfrak{p}$ is called *ramified* in X/Y (or in B/A). The natural morphism $\pi : X \longrightarrow Y$ (or the extension of rings B/A) is called unramified if X/Y (or B/A) is unramified in every point (prime ideal) of X (or B), [AG]. The point (prime ideal) $\mathfrak{p} \in Y$ is called *ramified* in X/Y (B/A) if $B_\mathfrak{p}/A_\mathfrak{p}$ is a ramified extension.

The extension B/A is unramified if and only if B is a separable A-algebra [AB, Theorem 2.5] (recall that B is supposed to be a finitely generated A-module), and the prime ideal $\mathfrak{p}$ is ramified in B/A if and only if $\mathfrak{N}_{B/A} \subset \mathfrak{p}$ [AB, Theorem 2.7].

Put

$$X_r = \{\mathfrak{p} \in X \quad | \quad \mathfrak{p} \quad \text{is ramified in} \quad X/Y\}.$$

Note that $X_r = V(\mathfrak{N}_{B/A})$ and therefore X_r is a closed subset of X. The set X_r is called the *branch locus* of X/Y.

4.2. **The branch locus over an invariant ring.** Now let $A = B^G$ where G is a finite group of automorphisms. Below we consider only cases where B is a finitely generated A-module (this always holds if B is a finitely generated algebra over a field K [Bour, V, 1.9. Theorem 2]).

The ring B is a separable A-algebra if and only if $I_\mathfrak{m} = 1$ for every maximal ideal $\mathfrak{m}$ of B [ChHR, Theorem 1.3]. Now let $\mathfrak{p} \in X = \text{Spec } B$, $\mathfrak{p}' = \mathfrak{p} \cap A$ and let $\mathfrak{p}_1 = \mathfrak{p}, \mathfrak{p}_2, \ldots, \mathfrak{p}_m$ be the set of prime ideals of B which lie over $\mathfrak{p}'$. Then the prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_m$ are in the same G-orbit [Bour, V, 2.2. Theorem 2]. Thus for every $i = 1, \ldots, m$ we have $I_{\mathfrak{p}_i} = g_i I_\mathfrak{p} g_i^{-1}$ for some $g_i \in G$. Further, consider the localization $B_{\mathfrak{p}'}$. The ring $B_{\mathfrak{p}'}$ is semilocal and the set of maximal ideals of $B_{\mathfrak{p}'}$ consists of images $\tilde{\mathfrak{p}}_1, \ldots, \tilde{\mathfrak{p}}_m$ of $\mathfrak{p}_1, \ldots, \mathfrak{p}_m$ (that is, $\tilde{\mathfrak{p}}_i = \mathfrak{p}_i \otimes_A A_{\mathfrak{p}'}$). Obviously, the inertia subgroup $I_{\tilde{\mathfrak{p}}_i}$ of the extension $B_{\mathfrak{p}'}/A_{\mathfrak{p}'}$ coincides with $I_{\mathfrak{p}_i}$. Moreover, $B_{\mathfrak{p}'}^G = A_{\mathfrak{p}'}$ [Bour, V,1.9. Proposition 23]. Thus $B_{\mathfrak{p}'}$ is a separable $A_{\mathfrak{p}'}$- algebra if and only if $I_{\mathfrak{p}_1} = 1$. This implies, in its turn, that the ideal $\mathfrak{p}_1$ is unramified in B/A if and only if $I_{\mathfrak{p}_1} = 1$ (see 4.1). Hence $X_r = \{\mathfrak{p} \in X \quad | \quad I_\mathfrak{p} \neq 1\}$. Now let $\{X_r^j\}_j$ be the set of irreducible components of X_r and let

$$\mathfrak{P}^1 = \{\mathfrak{p} \in X_r \quad | \quad \text{for } \mathfrak{q} \in X, \ \mathfrak{q} \subsetneqq \mathfrak{p} \ \text{ we have } \ I_\mathfrak{q} = 1\}.$$

The definition of $\mathfrak{P}^1$ obviously implies that for every j there exists a prime ideal $\mathfrak{p} \in \mathfrak{P}^1$ such that $V(\mathfrak{p}) = X_r^j$. Conversely, for every $\mathfrak{p} \in \mathfrak{P}^1$ the set $V(\mathfrak{p})$ coincides with an irreducible component of X_r.

4.3. **Wild ramification.** Let $pB = 0$ for some prime p. We say that the point (prime ideal) $\mathfrak{p} \in X$ (resp. $\mathfrak{p} \subset B$) is *wildly ramified* in X/Y (resp. in B/A) if $p \mid |I_{\mathfrak{p}}|$. The set of all wildly ramified points in X is written as X_{wr}. B/A is said to be wildly ramified if $X_{wr} \neq \emptyset$.

Let

$$\mathfrak{P}^1_w = \{\mathfrak{p} \in X_{wr} \mid \text{if } \mathfrak{q} \subsetneqq \mathfrak{p} \text{ then } \mathfrak{q} \notin X_{wr}\}.$$

Obviously,

$$X_{wr} = \bigcup_{\mathfrak{p} \in \mathfrak{P}^1_w} V(\mathfrak{p}).$$

Moreover, the definitions of $\mathfrak{P}^1, X_{wr}, I_{\mathfrak{p}}$ imply that $\mathfrak{P}^1$ is a finite set, X_{wr} is the closed subset of X and $\{V(\mathfrak{p})\}_{\mathfrak{p} \in \mathfrak{P}^1}$ is the set of irrducible components of X_{wr}.

4.4. **The total branch locus.** Again we assume that $A = B^G$, the group G is finite, and B is finitely generated as an A-module. Put

$$\tilde{B} = \prod_{H \le G} B^H \quad \text{and} \quad \tilde{X} = \text{Spec } \tilde{B},$$

$$\tilde{X}_r = \{\mathfrak{p} \in \tilde{X} \mid \mathfrak{p} \text{ is ramified in } \tilde{X}/Y\}.$$

The affine scheme $\tilde{X}$ is the disjoint union of irreducible components which can be identified with Spec B^H. The set $\tilde{X}_r = V(\mathfrak{N}_{\tilde{B}/A})$ is a closed subset of $\tilde{X}$. We call the set $\tilde{X}_r$, which is the branch locus of $\tilde{X}/Y$, also the *total branch locus* of X/Y.

We define:

$$\mathfrak{P}^0 = \{\mathfrak{p} \in X_r \mid I_{\mathfrak{p}} \neq \langle I_{\mathfrak{q}} \mid \mathfrak{q} \in X, \mathfrak{q} \subsetneqq \mathfrak{p}\rangle\}.$$

We clearly have the inclusions $\mathfrak{P}^1 \subseteq \mathfrak{P}^0$.

Remark 4.1. Notice that in the case $B = S(V)$, $G \le GL(V)$, $|G| = p^k$, $p = \text{char } K$ (where K is a ground field) a prime ideal $\mathfrak{p} \in \text{Spec } B$ lies in $\mathfrak{P}^0$ if and only if $I_{\mathfrak{p}}$ is shallow in the sense of [CHKSW]. In this case the depth of $B^{I_{\mathfrak{p}}}$ is equal to $\min\{\dim(V^{I_{\mathfrak{p}}}) + 2, \dim(V)\}$ [CHKSW].

Let $\{\tilde{X}_r^i\}_i$ be the set of irreducible components of $\tilde{X}_r$.

Proposition 4.2. *Suppose B is a normal geometric ring. Then for every i there exists a prime ideal $\mathfrak{p} \in \mathfrak{P}^0$ and a subgroup $H \le G$ such that*

$$V(\mathfrak{p}') = \tilde{X}_r^i,$$

where $\mathfrak{p}' = \mathfrak{p} \cap B^H$ *(here we consider* $V(\mathfrak{p}')$ *as a closed subset of* Spec B^H*; recall that we identify irreducible components of* $\tilde{X}$ *with sets of the form* Spec B^H*).*

Conversely, for every $\mathfrak{p} \in \mathfrak{P}^0$ *there exists a subgroup* $H \leq G$ *such that* $V(\mathfrak{p}')$ *coincides with an irreducible component of* $\tilde{X}_r$ *(here* $\mathfrak{p}' = \mathfrak{p} \cap B^H$ *and* $V(\mathfrak{p}') \subset$ Spec B^H*).*

Proof. We need the following lemmas.

Lemma 4.3. *Let* E/F *be an extension of Noetherian rings such that* E *is a finitely generated* F*-module. Further, let* $\mathfrak{q} \in$ Spec E*,* $\mathfrak{q}' = \mathfrak{q} \cap F$*. The ideal* $\mathfrak{q}$ *is unramified in* E/F *if and only if* $\mathfrak{q}\widehat{E}_{\mathfrak{q}}$ *is unramified in* $\widehat{E}_{\mathfrak{q}}/\widehat{F}_{\mathfrak{q}'}$*.*

Proof. This can be checked by routine considerations of residue fields and maximal ideals of the corresponding rings and their completions. □

Lemma 4.4. *Let* $F \subset T \subset E$ *be normal geometric rings such that* E *is finitely generated as an* F*-module. Further, let* $\mathfrak{p} \in$ Spec E*,* $\mathfrak{q} = \mathfrak{p} \cap T$*, and* $\mathfrak{r} = \mathfrak{q} \cap F$*. Then the ideal* $\mathfrak{p}$ *is unramified in* E/F *if and only if the ideal* $\mathfrak{p}$ *is unramified in* E/T *and the ideal* $\mathfrak{q}$ *is unramified in* T/F*.*

Proof. Lemma 4.3 implies that the assertion of the lemma can be reduced for the case of complete local rings $\widehat{F}_{\mathfrak{r}} \subset \widehat{T}_{\mathfrak{q}} \subset \widehat{E}_{\mathfrak{p}}$. Note that all rings in this sequence are normal because they are completions of normal geometric rings [N, 37.5]. Moreover, $\widehat{E}_{\mathfrak{p}}$ is finitely generated as an $\widehat{F}_{\mathfrak{r}}$-module. Further, if S/R is an extension of Noetherian rings such that S is a local ring and a finitely generated R-module, then this extension is unramified if and only if the maximal ideal of S is unramified in S/R [AB, Theorem 2.5]. Since all rings in the sequence $\widehat{F}_{\mathfrak{r}} \subset \widehat{T}_{\mathfrak{q}} \subset \widehat{E}_{\mathfrak{p}}$ are local and normal the assertion follows from [AB, Theorem A.2]. □

In the next lemma we maintain the notation of the Proposition.

Lemma 4.5. *Let* $H \leq G$ *and let* $\mathfrak{p} \in X$*,* $\mathfrak{q} = \mathfrak{p} \cap B^H$*. Then the ideal* $\mathfrak{q}$ *is unramified in* B^H/A *if and only if* $I_{\mathfrak{p}} \leq H$*.*

Proof. Suppose that $I_{\mathfrak{p}} \leq H$. Put $\mathfrak{r} = \mathfrak{p} \cap B^{I_{\mathfrak{p}}}$. The ideal $\mathfrak{r}$ is unramified in $B^{I_{\mathfrak{p}}}/A$ [N, 41.2]. Hence $\mathfrak{q}$ is unramified in B^H/A by Lemma 4.4.

Suppose $H_0 = I_{\mathfrak{p}} \cap H \neq I_{\mathfrak{p}}$. Put $\mathfrak{s} = \mathfrak{p} \cap B^{H_0}$. The ideal $\mathfrak{s}$ is unramified in B^{H_0}/B^H [N, 41.2]. Assume that $\mathfrak{q}$ is unramified in B^H/A. Then, by Lemma 4.4, $\mathfrak{s}$ is unramified in B^{H_0}/A and therefore (again by Lemma 4.4) the ideal $\mathfrak{s}$ is unramified in $B^{H_0}/B^{G_{\mathfrak{p}}}$. Then the degree of the maximal separable extension of $A_{\mathfrak{p}'}/\mathfrak{p}'A_{\mathfrak{p}'}$ (where $\mathfrak{p}' = \mathfrak{p} \cap A$) which is contained in $B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}}$ is greater that or equal to $[G_{\mathfrak{p}} : H_0]$ and, therefore, this degree exceeds $[G_{\mathfrak{p}} : I_{\mathfrak{p}}]$. But it should be equal to $[G_{\mathfrak{p}} : I_{\mathfrak{p}}]$ [Bour, V. 2.2, Proposition 5]. Hence we get a contradiction and therefore $\mathfrak{q}$ is ramified in B^H/A. □

Now we return to the proof of the Proposition.

The definition of $\tilde{X}$ implies that every closed irreducible subset of $\tilde{X}$ has the form $V(\mathfrak{p}') \subset \operatorname{Spec} B^H$ for some $H \leq G$ and some prime ideal $\mathfrak{p}'$ of B^H. Now let $\tilde{X}_r^i = V(\mathfrak{p}')$ and let $\mathfrak{p}$ be a prime ideal of B such that $\mathfrak{p}' = \mathfrak{p} \cap B^H$. We have to show that $\mathfrak{p} \in \mathfrak{P}^0$. The definition of the ideal $\mathfrak{p}'$ implies that $\mathfrak{p}'$ is ramified in B^H/A. Thus $I_\mathfrak{p} \not\subset H$ by Lemma 4.5. Suppose $I_\mathfrak{q} \not\subset H$ for some $\mathfrak{q} \subsetneqq \mathfrak{p}$. Then the ideal $\mathfrak{q}' = \mathfrak{q} \cap B^H$ is ramified in B^H/A (by Lemma 4.5) and therefore the closed irreducible subset $V(\mathfrak{q}')$ of Spec B^H is contained in $\tilde{X}_r$. But $V(\mathfrak{p}') \subsetneqq V(\mathfrak{q}')$. This is a contradiction to the choice of $\mathfrak{p}'$. Thus $I_\mathfrak{q} \subseteq H$ for every $\mathfrak{q} \subsetneqq \mathfrak{p}$ and therefore $I_\mathfrak{p} \neq \langle I_\mathfrak{q} \mid \mathfrak{q} \subsetneqq \mathfrak{p} \rangle$. Thus $\mathfrak{p} \in \mathfrak{P}^0$.

Now let $\mathfrak{p} \in \mathfrak{P}^0$. Put $H = \langle I_\mathfrak{q} \mid \mathfrak{q} \subsetneqq \mathfrak{p} \rangle$. Let $\mathfrak{p}' = \mathfrak{p} \cap B^H$. Since $I_\mathfrak{p} \not\subset H$, the prime ideal $\mathfrak{p}'$ is ramified in B^H/A and therefore the closed irreducible subset $V(\mathfrak{p}')$ of Spec B^H belongs to $\tilde{X}_r$. The definition of H implies that for every prime ideal $\mathfrak{q} \subsetneqq \mathfrak{p}$ the corresponding ideal $\mathfrak{q}' = \mathfrak{q} \cap B^H$ is unramified in B^H/A. Thus the set $V(\mathfrak{p}')$ coincides with an irreducible component of $\tilde{X}$. □

Define

$$c_s(\tilde{X}_r) = \sup \{\operatorname{codim} \tilde{X}_r^i\}_i.$$

Proposition 4.6. *If B is a geometric normal ring then*

$$c_s(\tilde{X}_r) = \sup \{\operatorname{ht}(\mathfrak{p}) \mid \mathfrak{p} \in \mathfrak{P}^0\}.$$

Proof. This follows from Proposition 4.2. □

5. The Cohen-Macaulay defect for actions on Noetherian rings

Let A be a commutative Noetherian ring. We define the Cohen-Macaulay defect as

$$\operatorname{cmdef} A := \sup\{\dim A_\mathfrak{p} - \operatorname{depth} A_\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Spec} A\}.$$

If A is a geometric ring we define

$$\operatorname{depth} A := \inf\{\operatorname{depth} A_\mathfrak{m} \mid \mathfrak{m} \subset A \text{ maximal ideal}\}.$$

Remark 5.1. A few remarks should be made why these definitions make sense.

(1) Clearly A is Cohen-Macaulay if and only if $\operatorname{cmdef} A = 0$.
(2) Lemma 5.2 below shows that we only need to consider maximal ideals of A in the definition of $\operatorname{cmdef} A$. Hence if A is a geometric ring, then $\operatorname{cmdef} A = \dim A - \operatorname{depth} A$.
(3) If A is a graded domain with a field as the degree-0 part or a Noetherian local ring, then

$$\operatorname{cmdef} A = \dim A - \operatorname{depth} A,$$

where $\operatorname{depth} A$ is defined (as usual) to be $\operatorname{grade}(\mathfrak{m}, A)$ with $\mathfrak{m}$ the maximal (homogeneous) ideal (see Lemma 5.2 below for A local and [K4, Satz 5.4] for A graded).

Lemma 5.2. *Let A be a Noetherian ring and $\mathfrak{p}, \mathfrak{q} \in \operatorname{Spec} A$ with $\mathfrak{p} \subseteq \mathfrak{q}$. Then*

$$\dim A_{\mathfrak{q}} - \operatorname{depth} A_{\mathfrak{q}} \geq \dim A_{\mathfrak{p}} - \operatorname{depth} A_{\mathfrak{p}}.$$

Proof. We need to show the following: If R is a Noetherian local ring and $\mathfrak{p} \in \operatorname{Spec} R$, then

$$\dim R - \operatorname{depth} R \geq \dim R_{\mathfrak{p}} - \operatorname{depth} R_{\mathfrak{p}}.$$

But this follows directly from the inequality

$$\operatorname{depth} R \leq \operatorname{depth} R_{\mathfrak{p}} + \dim R/\mathfrak{p} \tag{1}$$

(see [VW, Proposition 3.4]). □

Lemma 5.3. *Let B a Noetherian commutative ring, G a finite group of automorphisms of B and $A = B^G$. For a prime ideal $\mathfrak{p} \in \operatorname{Spec} B$ set $\mathfrak{q} := A \cap \mathfrak{p} \in \operatorname{Spec} A$. Then $\operatorname{ht}(\mathfrak{q}) = \operatorname{ht}(\mathfrak{p})$.*

Proof. Take a chain

$$\mathfrak{p}_0 \subsetneqq \mathfrak{p}_1 \subsetneqq \cdots \subsetneqq \mathfrak{p}_k = \mathfrak{p}$$

of prime ideals in B. Then $\mathfrak{q}_i := A \cap \mathfrak{p}_i$ gives an increasing chain in $\operatorname{Spec} A$, and the inclusions are strict by [Bour, V., 2.1., Corollary 1]. Thus $\operatorname{ht}(\mathfrak{q}) \geq \operatorname{ht}(\mathfrak{p})$.

For the reverse inequality, take a chain

$$\mathfrak{q}_0 \subsetneqq \mathfrak{q}_1 \subsetneqq \cdots \subsetneqq \mathfrak{q}_k = \mathfrak{q}$$

with $\mathfrak{q}_i \in \operatorname{Spec} A$. Starting with $\mathfrak{p}_k = \mathfrak{p}$, we construct a descending chain of prime ideals $\mathfrak{p}_i \in \operatorname{Spec} B$ with $A \cap \mathfrak{p}_i = \mathfrak{q}_i$. Suppose $\mathfrak{p}_i$ has already been constructed. By [Bour, V., 2.1., Theorem 1 and Corollary 2] there exists $\mathfrak{p}'_{i-1} \in \operatorname{Spec} B$ with $A \cap \mathfrak{p}'_{i-1} = \mathfrak{q}_{i-1}$, and there exists $\mathfrak{p}'_i \in \operatorname{Spec} B$ such that $A \cap \mathfrak{p}'_i = \mathfrak{q}_i$ and $\mathfrak{p}'_{i-1} \subseteq \mathfrak{p}'_i$. Hence $\mathfrak{p}'_i$ and $\mathfrak{p}_i$ both lie over $\mathfrak{q}_i$. By [Bour, V., 2.2., Theorem 2] this implies $\mathfrak{p}_i = g(\mathfrak{p}'_i)$ with $g \in G$. Now we take $\mathfrak{p}_{i-1} := g\left(\mathfrak{p}'_{i-1}\right)$ and obtain $A \cap \mathfrak{p}_{i-1} = \mathfrak{q}_{i-1}$ and $\mathfrak{p}_{i-1} \subseteq \mathfrak{p}_i$, as desired. The inclusion is strict since this holds for the inclusion of $\mathfrak{q}_{i-1}$ in $\mathfrak{q}_i$. □

The following lemma is known and the statement holds in a more general situation (see [LP]).

Lemma 5.4. *Let B be a commutative ring, G a finite group of automorphisms of B and $N \trianglelefteq G$ a normal subgroup. Then for $\mathfrak{p} \in \operatorname{Spec} B$ we have*

$$tr_{G/N}(B^N) \subseteq \mathfrak{p} \iff [I_{\mathfrak{p}} : N \cap I_{\mathfrak{p}}] \in \mathfrak{p}.$$

Proof. [LP, Lemma 1.1]. □

5.1. **Reflections.** Let g be an automorphism of a Noetherian commutative ring. We say that g is a k-reflection (for a non-negative integer k) if there exists a $\mathfrak{p} \in \operatorname{Spec} B$ with $\operatorname{ht}(\mathfrak{p}) \leq k$ such that $g \in I_{\mathfrak{p}}$. Thus in the special case where V is a finite-dimensional vector space, $g \in \operatorname{GL}(V)$, and $B = S(V)$, g is a 1-reflection if it is the identity or a pseudo-reflection in the usual sense.

Theorem 5.5. *Let B be a Noetherian Cohen-Macaulay ring containing the prime field F_p, and let G be a finite group of automorphisms of B. Suppose $A := B^G$ is Noetherian and set $k := \operatorname{cmdef}(A) + 2$. Let $N \leq G$ be the subgroup generated by all k-reflections and all elements of order not divisible by p, and assume that B^N is also Noetherian. Then B^N/A is unramified. In particular, if $G = I_{\mathfrak{p}}$ for a $\mathfrak{p} \in \operatorname{Spec} B$, then $N = G$.*

Proof. Assume that B^N/A is ramified. Observe that N is normal in G. Since $A = \left(B^N\right)^{G/N}$ and G/N is a p-group, Lemma 5.4 tells us that $I := tr_{G/N}(B^N) \neq A$. This means that $\widehat{H}^0\left(G/N, B^N\right) \neq 0$ (see Section 2.4), hence $\widehat{H}^1\left(G/N, B^N\right) \neq 0$ by (3.3). Pick a $\beta \in \widehat{H}^1\left(G/N, B^N\right) \setminus \{0\}$. The inflation map $\inf : \widehat{H}^1\left(G/N, B^N\right) \to \widehat{H}^1(G, B)$ is injective (3.2), hence $\alpha := \inf(\beta) \neq 0$. By [K3, Corollary 2.4], $I \subseteq \operatorname{Ann}_A(\alpha)$, and hence Corollary 1.6 in [K1] (which uses the hypothesis that B is Cohen-Macaulay) yields

$$\operatorname{grade}(I, A) \leq 2.$$

Here the grade means the maximal length of an A-regular sequence with elements in I. By [BH, Proposition 1.2.10.(a)] there exists a $\mathfrak{q} \in \operatorname{Spec} A$ with $I \subseteq \mathfrak{q}$ such that $\operatorname{depth} A_{\mathfrak{q}} = \operatorname{grade}(I, A)$. Thus

$$\operatorname{depth} A_{\mathfrak{q}} \leq 2. \tag{2}$$

Let $\mathfrak{p} \in \operatorname{Spec} B$ be a prime ideal lying over $\mathfrak{q}$. Then $I \subseteq \mathfrak{p}$, and since I is the relative trace ideal we obtain by Lemma 5.4 that there exists a $g \in I_{\mathfrak{p}} \setminus N$. By the definition of N we obtain $\operatorname{ht}(\mathfrak{p}) > k$. Thus $\operatorname{ht}(\mathfrak{q}) > k$ by Lemma 5.3. Combining this with (2) yields

$$\dim A_{\mathfrak{q}} - \operatorname{depth} A_{\mathfrak{q}} > k - 2 = \operatorname{cmdef} A,$$

a contradiction.

Hence B^N/A must be unramified. If in addition $G = I_{\mathfrak{p}}$ for $\mathfrak{p} \in \operatorname{Spec} B$, then $N = G$ by Lemma 4.5. $\square$

Remark 5.6. Since B^N/A is unramified if and only if the relative trace map $tr_{G/N}$ is surjective (Lemma 5.4), Theorem 5.5 is a generalization of Corollary 4.3 in [LP].

For linear actions of p-groups the following corollary is a direct consequence of Theorem 5.5.

Corollary 5.7. *If $G \leq \mathrm{GL}(V)$ is a p-group acting linearly on a finite-dimensional vector space V over a field K of characteristic p and $B = S(V)$, then G is generated by k-reflections, where $k = \dim V - \operatorname{depth} B^G + 2$.*

5.2. **A bound on the Cohen-Macaulay defect and vector invariants.**

Proposition 5.8. *Let B be a commutative ring containing F_p, let G be a finite group of automorphisms of B and write $A = B^G$. If B/A is wildly ramified, then there exists an integer r with $0 < r < 2|G_p|$, where $G_p \leq G$ is a Sylow p-subgroup, such that*

$$\widehat{H}^r(G, B) \neq 0.$$

Proof. From the wild ramification we have a $\mathfrak{p} \in \operatorname{Spec} B$ such that p divides $|I_\mathfrak{p}|$, so there exists a cyclic subgroup $H \subseteq I_\mathfrak{p}$ of order p. By (3.4) there exists a positive integer $r < 2|G_p|$ and an $\alpha \in \widehat{H}^r(G, F_p)$ such that $\operatorname{res}(\alpha) \in \widehat{H}^r(H, F_p)$ is non-zero. The composition of the natural maps $F_p \to B \to B/\mathfrak{p}$ is an injective $F_p[H]$-homomorphism. Since $B/\mathfrak{p}$ is trivial as a $F_p[H]$-module, the composition induces an injective map $\widehat{H}^r(H, F_p) \to \widehat{H}^r(H, B/\mathfrak{p})$. Therefore the natural map $\varphi : \widehat{H}^r(H, F_p) \to \widehat{H}^r(H, B)$ is also injective. Hence $\varphi(\operatorname{res}(\alpha)) \neq 0$, but this is the same as first mapping α into $\widehat{H}^r(G, B)$ and then restricting to H. The result follows. $\square$

For a non-empty Zariski-closed subset $Z \subseteq X := \operatorname{Spec} B$ we write $\operatorname{ht}(Z) := \inf\{\operatorname{ht}(\mathfrak{p}) \mid \mathfrak{p} \in Z\}$. If Z is empty we set $\operatorname{ht}(Z) = -\infty$.

Theorem 5.9. *Let B be a Noetherian Cohen-Macaulay ring containing F_p and let G be a finite group of automorphisms of B. Assume that $A := B^G$ is Noetherian and r is a positive integer with $\widehat{H}^r(G, B) \neq 0$. Then*

$$\operatorname{cmdef} A \geq \operatorname{ht}(X_{wr}) - r - 1,$$

where X_{wr} is the wild ramification locus in $X = \operatorname{Spec} B$. In particular,

$$\operatorname{cmdef} A \geq \operatorname{ht}(X_{wr}) - 2|G_p|,$$

where $G_p \leq G$ is a Sylow p-subgroup.

Proof. We may assume that r is minimal positive with $\widehat{H}^r(G, B) \neq 0$. Choose a non-zero element $\alpha \in \widehat{H}^r(G, B)$. Corollary 2.4 in [K3] says that $I := tr(B) \subseteq \operatorname{Ann}_A(\alpha)$. By [K1, Corollary 1.6] we have

$$\operatorname{grade}(I, A) \leq r + 1.$$

I is a proper ideal, thus by [BH, Proposition 1.2.10(a)] there exists $\mathfrak{q} \in \operatorname{Spec} A$ with $I \subseteq \mathfrak{q}$ and $\operatorname{grade}(I, A) = \operatorname{depth} A_\mathfrak{q}$. We obtain

$$\operatorname{cmdef} A \geq \dim A_\mathfrak{q} - \operatorname{depth} A_\mathfrak{q} \geq \operatorname{ht}(\mathfrak{q}) - r - 1. \tag{3}$$

Choose a $\mathfrak{p} \in \operatorname{Spec} B$ with $A \cap \mathfrak{p} = \mathfrak{q}$. Then $\operatorname{ht}(\mathfrak{p}) = \operatorname{ht}(\mathfrak{q})$ by Lemma 5.3. But $I \subseteq \mathfrak{p}$ implies $\mathfrak{p} \in X_{wr}$ (see Lemma 5.4), hence $\operatorname{ht}(\mathfrak{p}) \geq \operatorname{ht}(X_{wr})$. Putting this together with (3) yields the first result. The second result now follows from Proposition 5.8. □

We want to get a special version of Theorem 5.9 which does not use the somewhat technical concepts of the Cohen-Macaulay defect and height. We need the following Lemma, which is a compilation of [Bour, V, 1.9, Theorem 2 and Proposition 23].

Lemma 5.10. *Let S be a finitely generated algebra over a Noetherian ring R and let $B = S_U$ be the localization with respect to a multiplicative subset $U \subset S$. Suppose that G is a finite group of automorphisms of B which fix the image of R in B pointwise. Then $A := B^G$ can also be obtained by localizing a finitely generated R-algebra. Moreover, B is finitely generated as an A-module.*

In the above lemma A is Noetherian. However, it is not always true that the invariant ring R^G under a finite group of automorphisms of a Noetherian ring R is also Noetherian. For counter examples, see [M, Example 5.5].

Lemma 5.11. *Let B be a geometric ring over K and let G be a finite group of K-automorphisms of B. Then $A := B^G$ is also geometric and B is finitely generated as an A-module.*

Proof. We only have to show that A is equidimensional since everything else is already contained in Lemma 5.10. So let $\mathfrak{m}_1$ and $\mathfrak{m}_2$ be two maximal ideals in A. Choose $\mathfrak{m}_i' \in \operatorname{Spec} B$ lying over $\mathfrak{m}_i$. Then the $\mathfrak{m}_i'$ are maximal, hence $\operatorname{ht}(\mathfrak{m}_1') = \operatorname{ht}(\mathfrak{m}_2')$. By Lemma 5.3 it follows that $\mathfrak{m}_1$ and $\mathfrak{m}_2$ have the same height. □

Corollary 5.12. *Let B be a geometric Cohen-Macaulay ring over a field K of characteristic p and let G be a finite group of K-automorphisms of B. If the wild ramification locus X_{wr} in $X := \operatorname{Spec} B$ is non-empty, then*

$$\operatorname{depth} B^G \leq \dim(X_{wr}) + 2|G_p|,$$

where $G_p \leq G$ is a Sylow p-subgroup.

Proof. From Theorem 5.9 we have $\operatorname{cmdef} A \geq \operatorname{ht}(X_{wr}) - 2|G_p|$, where $A = B^G$. By Lemma 5.11 and Remark 5.1(b) we have $\operatorname{cmdef} A = \dim A - \operatorname{depth} A$, hence

$$\operatorname{depth} A \leq \dim A - \operatorname{ht}(X_{wr}) + 2|G_p|. \tag{4}$$

Since B is catenary (i.e., all maximal chains of prime ideals between two given prime ideals have equal length) and all maximal ideals have the same height we have $\dim(B/I) + \operatorname{ht}(I) = \dim(B)$ for any proper ideal I in B. In particular, $\operatorname{ht}(X_{wr}) = \dim B - \dim X_{wr}$. Moreover, $\dim A = \dim B$ since B is finite as an A-module. Combining this with (4) yields the required bound. □

If G acts linearly on a vector space V, one can (and often does) consider invariants of the G-action on the n-fold direct sum V^n of V. The following corollary is about a slightly more general situation, where the action of G need not be linear. Observe that $S(V^n) \cong S(V)^{\otimes n}$.

Corollary 5.13. *Let B be a polynomial ring over a field K and let G be a finite group of K-automorphisms of B. If B/B^G is wildly ramified, then*

$$\lim_{n\to\infty} \operatorname{cmdef} \left(B^{\otimes n}\right)^G = \infty.$$

Here G acts diagonally on the tensor power.

Remark 5.14. If in the situation of Corollary 5.13 B/B^G is not wildly ramified, then $B^{\otimes n}/\left(B^{\otimes n}\right)^G$ is not wildly ramified, either, and it follows from [K3, Theorem 2.10] and [K1, Theorem 1.4] that $\left(B^{\otimes n}\right)^G$ is Cohen-Macaulay, so in this case

$$\operatorname{cmdef} \left(B^{\otimes n}\right)^G = 0$$

for every n. An example of this type is given by an action generated by a translation which maps some indeterminate x_i of B to $x_i + 1$.

Proof of Corollary 5.13. Corollary 5.12 yields

$$\operatorname{cmdef} \left(B^{\otimes n}\right)^G \geq nm - \dim\left(X^n_{wr}\right) - 2|G_p|, \tag{5}$$

where $m = \dim B$, $X^n = \operatorname{Spec} B^{\otimes n}$, and $G_p \leq G$ is a Sylow p-subgroup. Consider the algebraic set $Y \subseteq \bar{K}^m$ (with $\bar{K}$ an algebraic closure of K) given by the ideal in B defining the wild ramification locus X_{wr}. For $x \in \bar{K}^m$ we have $x \in Y$ if and only if p divides $|G_x|$. The variety $Y_n \subseteq (\bar{K}^m)^n$ corresponding to X^n_{wr} is clearly contained in Y^n, the n-fold cartesian product of Y. Thus $\dim Y_n \leq n \dim Y$. It follows that

$$\dim X^n_{wr} = \dim Y_n \leq n \dim Y. \tag{6}$$

But $\dim Y < m$ since no element $\sigma \in G$ of order p fixes all of $\bar{K}^m$. Substituting (6) into (5) yields the theorem. ☐

Corollary 5.15. *Let G be a finite group acting faithfully and linearly on a finite dimensional vector space V over a field of characteristic p, which divides $|G|$. Then*

$$\lim_{n\to\infty} \operatorname{cmdef} S(V^n)^G = \infty.$$

6. The depth of invariants of normal geometric rings with respect to a p-group

In the case when B is a normal geometric ring and G is a p-group we have

Theorem 6.1. *Let B be a normal geometric Cohen-Macaulay ring,* char $B = p \neq 0$. *Further, let $B^G = A$ where $|G| = p^k$ for some k. Then*

$$\operatorname{depth} A \leq \dim A - c_s(\tilde{X}_r) + 2.$$

Remark 6.2. Recall $c_s(\tilde{X}_r) = \sup\{\text{codim } \tilde{X}_r^i\}_i$ where $\{\tilde{X}_r^i\}_i$ is the set of irreducible components of $\tilde{X}_r$. We can consider the set $\{X_r^j\}_j$ of irreducible components of X_r as a subset of $\{\tilde{X}_r^i\}_i$. Hence $\dim A - c_s(\tilde{X}_r) \leq \dim A - \inf\{\text{codim } X_r^j\}_j = \dim A - \text{ht}(X_r) = \dim X_r$. Note that in the case $\mid G \mid = p^k$ we have $X_r = X_{wr}$ and $r = \min\{q \mid \widehat{H}^q(G, B) \neq 0\} = 1$. Thus, in the case considered here, the inequality of Theorem 6.1 implies 5.9 and 5.12.

Proof of Theorem 6.1. We need the following

Lemma 6.3. *Let $\mathfrak{p} \in \mathfrak{P}^0$, and set $C = B^{G_\mathfrak{p}}$ and $\mathfrak{q} = \mathfrak{p} \cap C$. Suppose* $\text{ht}(\mathfrak{p}) \geq 2$. *Then*

$$\text{depth } C_\mathfrak{q} = 2.$$

Proof. Put $k = \text{cmdef } C_\mathfrak{q} + 2$. Let

$$N = \langle I_\mathfrak{r} \mid \mathfrak{r} \subseteq \mathfrak{p}, \ \text{ht}(\mathfrak{r}) \leq k \rangle.$$

Then N is a normal subgroup of $G_\mathfrak{p}$ and, according to Theorem 5.5, the extension $B_\mathfrak{p}^N/C_\mathfrak{q}$ is unramified. Thus $I_\mathfrak{p} = N$ by Lemma 4.5. Now the definition of $\mathfrak{P}^0$ implies $\text{ht}(\mathfrak{p}) \leq k$. Thus,

$$k = \dim C_\mathfrak{q} - \text{depth } C_\mathfrak{q} + 2 \geq \dim C_\mathfrak{q} = \text{ht}(\mathfrak{q})$$

and, therefore, depth $C_\mathfrak{q} \leq 2$. The reverse inequality follows since every normal Noetherian ring satisfies Serre's condition (S_2) (see [BH, Theorem 2.2.22]). $\square$

Lemma 6.4. *Let $\mathfrak{p} \in \mathfrak{P}^0$ and let $\mathfrak{p}' = \mathfrak{p} \cap A$. If* $\text{ht}(\mathfrak{p}') \geq 2$ *then*

$$\text{depth } A_{\mathfrak{p}'} = 2.$$

Proof. We maintain the notation of the proof of the previous Lemma.

Let us show

$$\widehat{C}_\mathfrak{q} = \widehat{A}_{\mathfrak{p}'}. \tag{7}$$

Note that $\widehat{C}_\mathfrak{q}$ is a finitely generated $\widehat{A}_{\mathfrak{p}'}$-module. Indeed, B is a finitely generated A-module [N, 10.16]. Hence $B \otimes_A \widehat{A}_{\mathfrak{p}'}$ is a finitely generated $\widehat{A}_{\mathfrak{p}'}$-module. But

$$B \otimes_A \widehat{A}_{\mathfrak{p}'} = \prod_{i=1}^{s} \widehat{B}_{\mathfrak{p}_i},$$

where $\mathfrak{p}_1 = \mathfrak{p}, \ldots, \mathfrak{p}_s$ are prime ideals of B which lie over $\mathfrak{p}'$ [N, 17.7]. Hence $\widehat{B}_\mathfrak{p}$ is a finitely generated $\widehat{A}_{\mathfrak{p}'}$-module. Since $\widehat{C}_\mathfrak{q}$ is an $\widehat{A}_{\mathfrak{p}'}$-submodule of the $\widehat{A}_{\mathfrak{p}'}$-module $\widehat{B}_\mathfrak{p}$, it is also finitely generated.

Further, the ideal $\mathfrak{q}$ is unramified in C/A [N, 41.2]. Hence, by Lemma 4.3, the extension $\widehat{C}_\mathfrak{q}/\widehat{A}_{\mathfrak{p}'}$ is unramified. Since, in addition, the $\widehat{A}_{\mathfrak{p}'}$-module $\widehat{C}_\mathfrak{q}$ is finitely generated, it is a free module [AB, Proposition 4.5]. Hence $\widehat{C}_\mathfrak{q}/\mathfrak{p}'\widehat{C}_\mathfrak{q}$ is a finite separable extension of $\widehat{A}_{\mathfrak{p}'}/\mathfrak{p}'\widehat{A}_{\mathfrak{p}'}$, and the dimension of this extension is equal to $\operatorname{rank}_{\widehat{A}_{\mathfrak{p}'}} \widehat{C}_{\mathfrak{q}}$. But this dimension is equal to 1 [N, 41.2] and, therefore, we have (7).

Further,

$$\text{(8)} \qquad \operatorname{depth} A_{\mathfrak{p}'} = \operatorname{depth} \widehat{A}_{\mathfrak{p}'}, \ \operatorname{depth} C_{\mathfrak{q}} = \operatorname{depth} \widehat{C}_{\mathfrak{q}}$$

[S, IV, A.4. Proposition 9].

Now the assertion of the Lemma follows from (7), (8) and the previous Lemma. $\square$

Now we can finish the proof of Theorem 6.1.

Since B is a normal geometric ring, the ring A and its localizations are also geometric by Lemma 5.11. Hence for every maximal ideal $\mathfrak{m}'$ of A, the ring $A_{\mathfrak{m}'}$ is a factor ring of a regular local ring. Thus, if $\mathfrak{p}' \subset \mathfrak{m}'$, we have

$$\text{(9)} \qquad \operatorname{cmdef} A_{\mathfrak{p}'} \ \leq \ \operatorname{cmdef} A_{\mathfrak{m}'}$$

(Lemma 5.2). Since $\dim A_{\mathfrak{m}'} = \dim A$ and $\dim A_{\mathfrak{p}'} = \operatorname{ht}(\mathfrak{p})$, the assertion of the Theorem follows from (9), Lemma 6.4 and Proposition 4.6. $\square$

Corollary 6.5. *Let the notation and assumptions of Theorem 6.1 hold. Then all inertia subgroups of G are generated by k-reflections, where $k = \dim A - \operatorname{depth} A + 2$.*

Proof. By way of contradiction, assume that there exists $\mathfrak{p} \in X$ such that $I_{\mathfrak{p}} \neq \langle I_{\mathfrak{q}} \mid I_{\mathfrak{q}} \subseteq I_{\mathfrak{p}}, \operatorname{ht}(\mathfrak{q}) \leq k \rangle$. We can choose $\mathfrak{p}$ minimal with this property. Then for $\mathfrak{q} \in X$ with $\mathfrak{q} \subsetneqq \mathfrak{p}$ we have $I_{\mathfrak{q}} = \langle I_{\mathfrak{q}'} \mid I_{\mathfrak{q}'} \subseteq I_{\mathfrak{q}}, \operatorname{ht}(\mathfrak{q}') \leq k \rangle$. Since $I_{\mathfrak{q}} \subseteq I_{\mathfrak{p}}$, this implies $\mathfrak{p} \in \mathfrak{P}^0$. Since $\operatorname{ht}(\mathfrak{p}) > k$, Theorem 6.1 leads to the contradiction

$$\operatorname{depth} A < \dim A - k + 2 = \operatorname{depth} A.$$

$\square$

Corollary 6.6. *Let the notation and assumptions of Theorem 6.1 hold. Suppose that A is a Cohen-Macaulay ring. Then all inertia subgroups of G are generated by 2-reflections.*

Proof. This is a special case of Corollary 6.5. $\square$

Corollary 6.7. *Let the notation and assumptions of Theorem 6.1 hold. Suppose that $B = S(V)$ is the symmetric algebra of a linear space V and suppose the ideal $(V)^2$ generated by squares of linear forms is G-invariant (in particular, this holds if the G-action can be induced by a linear action on V). Then the group G as well as all inertia subgroups are generated by elements $g \in G$ such that*

$$\dim (g-1)V (\operatorname{mod} (V)^2) \ \leq \ \dim A - \operatorname{depth} A + 2.$$

If, in addition, A is a Cohen-Macaulay algebra Then the group G as well as all inertia subgroups are generated by elements $g \in G$ such that

$$\dim(g-1)V(\mathrm{mod}(V)^2) \ \leq \ 2.$$

Proof. Let $\mathfrak{p}$ be a prime ideal of B of the height h. Then the image of the natural map $\mathfrak{p} \longrightarrow (V)/(V)^2$ is a linear subspace which has dimension $\leq h$. Also, note that in this case the whole group G is the inertia subgroup of (V). $\square$

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Nikolai Gordeev, Department of Mathematics of Russian State Pedagogical University, Moijka 48, Sankt Petersburg 191-186, Russia

E-mail address: `gordeev@pdmi.ras.ru`

Gregor Kemper, IWR, Universität Heidelberg, Im Neuenheimer Feld 368, 69 120 Heidelberg, Germany

E-mail address: `Gregor.Kemper@iwr.uni-heidelberg.de`