Invariants of Hopf Algebras

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Introduction

In the winter of 1999 I gave a series of lectures at Queen's university about some recent results concerning the Cohen-Macaulay property of invariants of Hopf algebras. Tony Geramita asked me to write up my notes for the Queen's Papers, and I happily took up his suggestion. Although this article focuses on the proof of one main theorem (Theorem 2.11 on page 12), it has some of the character of a survey article, since the results already appeared in my (German) habilitation thesis [10], and I am trying to explain concepts in more detail and give more background than in an original article.

This work originated in the study of the Cohen-Macaulay property of invariant rings of finite groups [11], which led to results about invariants of algebraic groups as well [12]. Actions of Hopf algebras are a natural generalization of group actions. Apart from group rings, Lie algebras (or, more precisely, their universal enveloping algebras) are an interesting incarnation of Hopf algebras, so everything that we say in this article will apply to invariants if Lie algebras as well. By a celebrated theorem of Hochster and Roberts [8], invariant rings of linearly reductive groups are always Cohen-Macaulay, i.e., they are free modules of finite rank over a subalgebra which is isomorphic to a polynomial ring. With the proper definition, this carries over to invariants of linearly reductive Hopf algebras (see Theorem 3.4 in this paper). For finite groups, Hochster and Roberts's result means that the invariant ring is Cohen-Macaulay if the characteristic of the ground field does not divide the group order (see Hochster and Eagon [7]). On the other hand, it has been generally observed that invariant rings of finite groups tend not to be Cohen-Macaulay if the characteristic divides the group order. However, it is still an open question to characterize exactly those groups and representations for which the invariant ring is Cohen-Macaulay.

In this article we prove that if a Hopf algebra is geometrically reductive but not linearly reductive, then there exists a representation whose invariant ring is not Cohen-Macaulay. This constitutes a partial converse to the (generalized) theorem of Hochster and Roberts. This means, for example, that for each of the classical algebraic groups in positive characteristic there exists a representation such that the invariant ring is not Cohen-Macaulay. Moreover, every finite group of order divisible by the characteristic of the ground field has a representation with non-Cohen-Macaulay invariant ring. This gives a partial answer to the question raised above. For (universal enveloping algebras of) Lie algebras, the main theorem says the following: Any non-zero, finite dimensional Lie algebra $\mathfrak{g}$ in positive characteristic has a finite dimensional representation V such that the invariant ring (i.e., the set of polynomials in the symmetric algebra $S(V)$ on which all elements of $\mathfrak{g}$ act as zero) is not Cohen-Macaulay.

In the first section of the paper we recall the concept of Hopf algebras and their invariants, and introduce cohomology. This is used as the basic tool in Section 2 to develop a criterion for regular sequences in the invariant ring. It turns out that in order to prove that the invariant ring is not Cohen-Macaulay, it suffices to find a non-zero element $\alpha \in H^1(\Lambda, S(V))$ such that the annihilator $\mathrm{Ann}_{S(V)^\Lambda}(\alpha)$ contains a sequence of length three which generates an ideal of height three in $S(V)$ as well as in $S(V^\Lambda)$. In Section 3 such a situation is manufactured under the assumption that we have a geometrically reductive but not linearly reductive Hopf algebra. This proves the main theorem. In order to apply the theorem, one must interpret the concept of linear and geometric reductivity for the various classes of objects which occur as Hopf algebras. This is done in the fourth section.

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1 Hopf Algebras

Suppose that a group G acts by automorphisms on an algebra R over a field K. Then R is a module over the group ring KG, and for $a, b \in R$ and $\sigma \in G$ we have $\sigma(ab) = \sigma(a)\sigma(b)$. Thus R is not an algebra over KG in the usual sense. Moreover, if U and V are KG-modules, then $U \otimes_K V$ is also a KG-module, where

$$a\sigma(u \otimes v) = a(\sigma(u) \otimes \sigma(v))$$

for $a \in K$, $\sigma \in G$, $u \in U$, and $v \in V$. This is quite unusual, since in general the tensor product of two modules over an algebra A does not have the structure of an A-module. **We make the convention here that throughout this article all tensor products are over K.** Also note that $\mathrm{Hom}_K(U, V)$ (and in particular the dual $V^* = \mathrm{Hom}_K(V, K)$) is again a KG-module. These "peculiarities" of the algebra KG are captured by the fact that KG is a Hopf algebra with antipode. We recall the definition of a Hopf algebra and then present a few examples, including group rings.

Definition 1.1. *(a) A* **Hopf algebra** *over a field K is an associative K-algebra Λ with unit and homomorphisms of algebras $\Delta\colon \Lambda \to \Lambda \otimes \Lambda$ ("***comultiplication***") and $\epsilon\colon \Lambda \to K$ ("***counit***") such that the diagrams*

$$\begin{tikzcd}
\Lambda \arrow[r, "\Delta"] \arrow[d, "\Delta"] & \Lambda \otimes \Lambda \arrow[d, "\mathrm{id} \otimes \Lambda"] \\
\Lambda \otimes \Lambda \arrow[r, "\Delta \otimes \mathrm{id}"] & \Lambda \otimes \Lambda \otimes \Lambda
\end{tikzcd}$$

and

$$\begin{tikzcd} & \Lambda \arrow[dl, "\cong"'] \arrow[d, "\Delta"] \arrow[dr, "\cong"] & \\ \Lambda \otimes K & \Lambda \otimes \Lambda \arrow[l, "\mathrm{id} \otimes \epsilon"'] \arrow[r, "\epsilon \otimes \mathrm{id}"] & K \otimes \Lambda \end{tikzcd}$$

commute.

(b) A Hopf algebra Λ *is said to be* **commutative** *if it is commutative as a* K*-algebra, and* **cocommutative** *if the diagram*

$$\begin{tikzcd} & \Lambda \arrow[dl, "\Delta"'] \arrow[dr, "\Delta"] & \\ \Lambda \otimes \Lambda \arrow[rr, "\mathrm{swap}"] & & \Lambda \otimes \Lambda \end{tikzcd}$$

commutes, where the lower map is $a \otimes b \mapsto b \otimes a$*.*

(c) An **antipode** *of a Hopf algebra* Λ *is a* K*-linear map* $\eta\colon \Lambda \to \Lambda$ *such that the diagram*

$$\begin{tikzcd} \Lambda \otimes \Lambda \arrow[dd, "\eta \otimes \mathrm{id}"] & \Lambda \arrow[l, "\Delta"'] \arrow[r, "\Delta"] \arrow[d, "\epsilon"'] & \Lambda \otimes \Lambda \arrow[dd, "\mathrm{id} \otimes \eta"'] \\ & K \arrow[d] & \\ \Lambda \otimes \Lambda \arrow[r, "\mathrm{mult}"] & \Lambda & \Lambda \otimes \Lambda \arrow[l, "\mathrm{mult}"'] \end{tikzcd}$$

commutes.

(d) A **module** *over a Hopf algebra* Λ *is a (left-)module over* Λ *as a ring.* K *becomes a* Λ*-module via the counit* ϵ*. If* U *and* V *are* Λ*-modules, then* $U \otimes V$ *has a natural structure as a* $(\Lambda \otimes \Lambda)$*-module (this is true for any* K*-algebra* Λ*), and becomes a* Λ*-module via the comultiplication* Δ*. More explicitly, if* $\Delta(\lambda) = \sum_{i=1}^{r} \mu_i \otimes \nu_i$ *for a* $\lambda \in \Lambda$*, then for* $u \in U$ *and* $v \in V$ *we have*

$$\lambda(u \otimes v) = \sum_{i=1}^{r} \mu_i u \otimes \nu_i v.$$

If V is a Λ-module, then the dual V^* has a natural structure as a module over the opposite algebra Λ^{op}. It can be shown that an antipode η always is an anti-homomorphism of algebras (see Sweedler [22, Proposition 4.0.1 (1)]). Therefore if Λ has an antipode η, V^* becomes a Λ-module via η. Now the natural isomorphism $\operatorname{Hom}_K(U, V) \cong U^* \otimes V$ makes $\operatorname{Hom}_K(U, V)$ into a Λ-module. More explicitly, if $\Delta(\lambda) = \sum_{i=1}^r \mu_i \otimes \nu_i$ for a $\lambda \in \Lambda$, then for $f \in \operatorname{Hom}_K(U, V)$ we have

$$\lambda f \colon U \to V, \;\; u \mapsto \sum_{i=1}^{r} \mu_i f(\eta(\nu_i)u).$$

(e) If V is a module over a Hopf algebra Λ, we write

$$V^\Lambda := \{v \in V \mid \lambda v = \epsilon(\lambda)v \;\forall \lambda \in \Lambda\}$$

and call this the space of **invariants** *of Λ (see Montgomery [13, p. 13]).*

(f) A **Λ-algebra** *is an associative, commutative K-algebra R with unit which has a structure of a Λ-module such that $1_R \in R^\Lambda$ (the unit is an invariant), and the multiplication $\mathrm{mult} \colon R \otimes R \to R$ is a homomorphism of Λ-modules. More explicitly, we demand that $\lambda(ab) = \sum_{i=1}^r (\mu_i a)(\nu_i b)$ for $a, b \in R$.*

We look at a few examples of Hopf algebras, which show that Hopf algebras provide an umbrella for various structures that are quite familiar.

Example 1.2. (a) Let G be a group, K a field, and $\Lambda := KG$ the group ring. Λ becomes a cocommutative Hopf algebra with $\Delta(\sigma) = \sigma \otimes \sigma$ and $\epsilon(\sigma) = 1$ for $\sigma \in G$. An antipode is given by $\eta(\sigma) = \sigma^{-1}$. We see that with these definitions tensor products and homomorphism spaces of Λ-modules are given structures as Λ-modules in the usual way. If V is a Λ-module, we sometimes write V^G for V^Λ, which is the fixed space under the G-action. Also notice that a Λ-algebra is nothing else than a K-algebra with an action of G by automorphisms fixing K.

(b) Let $\mathfrak{g}$ be a Lie algebra over a field K and $\Lambda := U(\mathfrak{g})$ the universal enveloping algebra. In order to make Λ into a Hopf algebra, it is enough to define Δ and ϵ on $\mathfrak{g}$, since Λ is generated by $\mathfrak{g}$ as a K-algebra. We set $\Delta(x) = x \otimes 1 + 1 \otimes x$ and $\epsilon(x) = 0$ for $x \in \mathfrak{g}$. An antipode is given by $\eta(x) = -x$.

A module over Λ is also a module over the Lie algebra $\mathfrak{g}$ by restricting the action, and conversely a $\mathfrak{g}$-module is endowed with a Λ-action by the universal property of the universal enveloping algebra. In more sophisticated language, the module categories of $\mathfrak{g}$ and Λ are equivalent. A vector $v \in V$ is invariant if and only if $xv = 0$ for all $x \in \mathfrak{g}$. We sometimes write $V^{\mathfrak{g}}$ for V^Λ. By the special form of the comultiplication, a Λ-algebra is a K-algebra with an action of $\mathfrak{g}$ by derivations.

(c) A commutative Hopf algebra Λ with antipode is nothing else than the coordinate ring of an affine group scheme $G = \operatorname{Spec}(\Lambda)$ (see, for example, Waterhouse [23, Theorem 1.4]). Λ is cocommutative if and only if G is abelian.

(d) The following example is closely related to the above: Let G be a finite group, K a field, and let $\Lambda := \mathrm{Fun}(G, K)$ be the algebra of functions from G into K with pointwise addition and multiplication. Define $\epsilon\colon \Lambda \to K$ by $\epsilon(f) = f(1)$, and $\Delta\colon \Lambda \to \Lambda \otimes \Lambda \cong \mathrm{Fun}(G \times G, K)$ by

$$\Delta(f)\colon G \times G \to K, \ (\sigma, \tau) \mapsto f(\sigma\tau).$$

This makes Λ a commutative Hopf algebra with an antipode given by $\eta(f)\colon G \to K, \ \sigma \mapsto f(\sigma^{-1})$. For $\sigma \in G$, denote by $\delta_\sigma \in \Lambda$ the function $\tau \mapsto \delta_{\sigma,\tau}$ (the Kronecker delta).

Let V be a Λ-module. Since the δ_σ are orthogonal idempotents, $V_\sigma := \delta_\sigma V$ yields a decomposition

$$V = \bigoplus_{\sigma \in G} V_\sigma.$$

Thus V has the structure of a G-graded vector space. Conversely, any G-graded vector space V becomes a Λ-module by setting $f \cdot v = f(\sigma)v$ for $f \in \Lambda$ and $v \in V_\sigma$. In other words, the module category of Λ is equivalent to the category of G-graded vector spaces. The invariant space of V is simply the graded component V_1 associated to the unit element of G. A Λ-algebra is a G-graded algebra.

(e) Let $\Lambda := \mathcal{P}^*$ be the Steenrod algebra over a finite field $K = \mathbb{F}_q$. We refer to Smith [19, Section 11.1] for the definition. $\mathcal{P}^*$ is generated by the Steenrod operations $\mathcal{P}^k$. Λ becomes a cocommutative Hopf algebra by $\Delta(\mathcal{P}^k) = \sum_{i=0}^{k} \mathcal{P}^i \otimes \mathcal{P}^{k-i}$ and $\epsilon(\mathcal{P}^k) = \delta_{k,0}$. However, Λ does not have an antipode. A Λ-algebra is exactly what Smith calls an algebra over the Steenrod algebra. $\triangleleft$

Proposition 1.3. *Let Λ be a Hopf algebra and R a Λ-algebra.*

(a) For $f \in R^\Lambda$, multiplication with f provides a homomorphism $R \to R$ of Λ-modules.

(b) R^Λ is a subalgebra of R.

Proof. (a) Take $f \in R^\Lambda$, $g \in R$, and $\lambda \in \Lambda$ with $\Delta(\lambda) = \sum_{i=1}^{r} \mu_i \otimes \nu_i$. Then

$$\lambda(fg) = \sum_{i=1}^{r} (\mu_i f)(\nu_i g) = \sum_{i=1}^{r} (\epsilon(\mu_i) f)(\nu_i g) = f \cdot \sum_{i=1}^{r} \epsilon(\mu_i)\nu_i g = f \cdot (\lambda g),$$

where the second diagram in Definition 1.1(a) was used.

(b) We have $1_R \in R^\Lambda$. Moreover, if $f, g \in R^\Lambda$ and $\lambda \in \Lambda$, then by (a)

$$\lambda(fg) = f \cdot (\lambda g) = f\epsilon(\lambda)g = \epsilon(\lambda)(fg),$$

hence $fg \in R^\Lambda$. $\square$

We will need the following lemma, whose proof was shown to me by Moss Sweedler.

Lemma 1.4. *Let Λ be a Hopf algebra with antipode and U, V Λ-modules. Then*

$$(\operatorname{Hom}_K(U,V))^\Lambda = \operatorname{Hom}_\Lambda(U,V).$$

Proof. Take $\varphi \in \operatorname{Hom}_\Lambda(U,V)$. Then for $\lambda \in \Lambda$ with $\Delta(\lambda) = \sum_{i=1}^r \mu_i \otimes \nu_i$ and $u \in U$ we have

$$(\lambda\varphi)(u) = \sum_{i=1}^r \mu_i \varphi(\eta(\nu_i)u) = \sum_{i=1}^r \mu_i \eta(\nu_i)\varphi(u) = \epsilon(\lambda)\varphi(u),$$

where the diagram in Definition 1.1(c) was used. Hence $\varphi \in \operatorname{Hom}_K(U,V)^\Lambda$.

The opposite inclusion is much harder. For $\lambda \in \Lambda$ with $\Delta(\lambda) = \sum_{i=1}^r \mu_i \otimes \nu_i$, set

$$\Delta(\mu_i) = \sum_{j=1}^s \alpha_{i,j} \otimes \beta_{i,j} \quad \text{and} \quad \Delta(\nu_i) = \sum_{k=1}^t \gamma_{i,k} \otimes \delta_{i,k}.$$

By coassociativity (the first diagram in Definition 1.1(a)) we have

$$\sum_{i,j} \alpha_{i,j} \otimes \beta_{i,j} \otimes \nu_i = (\Delta \otimes \mathrm{id})(\Delta(\lambda)) = (\mathrm{id} \otimes \Delta)(\Delta(\lambda)) = \sum_{i,k} \mu_i \otimes \gamma_{i,k} \otimes \delta_{i,k}.$$

For $\varphi \in \operatorname{Hom}_K(U,V)$ and $u \in U$, this identity gives rise to

$$\sum_{i,j} \alpha_{i,j}\varphi(\eta(\beta_{i,j})\nu_i u) = \sum_{i,k} \mu_i \varphi(\eta(\gamma_{i,k})\delta_{i,k}u) = \sum_i \mu_i \varphi(\epsilon(\nu_i)u) = \lambda\varphi(u).$$

If $\varphi \in \operatorname{Hom}_K(U,V)^\Lambda$, then

$$\sum_{j=1}^s \alpha_{i,j}\varphi(\eta(\beta_{i,j})\nu_i u) = (\mu_i\varphi)(\nu_i u) = \epsilon(\mu_i)\varphi(\nu_i u).$$

With the above, this yields

$$\lambda\varphi(u) = \sum_{i=1}^r \varphi(\epsilon(\mu_i)\nu_i u) = \varphi(\lambda u).$$

This shows that $\varphi \in \operatorname{Hom}_\Lambda(U,V)$. $\square$

We now introduce cohomology of Hopf algebras. Take a projective resolution

$$\cdots \xrightarrow{\partial_2} P_2 \xrightarrow{\partial_1} P_1 \xrightarrow{\partial_0} P_0 \longrightarrow K \longrightarrow 0$$

of K (as a Λ-module). For a Λ-module V, we obtain the complex

$$\operatorname{Hom}_\Lambda(P_0,V) \xrightarrow{\partial_0^*} \operatorname{Hom}_\Lambda(P_1,V) \xrightarrow{\partial_1^*} \operatorname{Hom}_\Lambda(P_2,V) \xrightarrow{\partial_2^*} \cdots,$$

with $\partial_i^*(f) = f \circ \partial_i$. The i-th cohomology is defined as

$$H^i(\Lambda,V) := \operatorname{Ext}_\Lambda^i(K,V) := \ker(\partial_i^*)/\operatorname{im}(\partial_{i-1}^*),$$

where we formally set $\partial_i^* = 0$ for $i < 0$. The cohomology $H^i(\Lambda,V)$ is independent of the choice of the projective resolution (see Benson [1, p. 29]). As an important tool we have the long exact sequence of cohomology (see Benson [1, Proposition 2.5.3(ii)]). For later use and to gain some experience in working with cohomology, we prove the following lemma.

Lemma 1.5. *If Λ is a Hopf algebra and V a Λ-module, then*

$$H^0(\Lambda, V) \cong V^\Lambda.$$

Proof. By the second diagram in Definition 1.1(a), ϵ is not the zero-map, hence we can choose a projective resolution which starts as

$$\cdots \longrightarrow P_1 \xrightarrow{\partial_0} \Lambda \xrightarrow{\epsilon} K \longrightarrow 0.$$

Hence $H^0(\Lambda, V) = \{\varphi \in \mathrm{Hom}_\Lambda(\Lambda, V) \mid \varphi \circ \partial_0 = 0\}$. We have an isomorphism of vector spaces

$$\mathrm{Hom}_\Lambda(\Lambda, V) \xrightarrow{\sim} V,\ \varphi \mapsto \varphi(1).$$

Assume that $\varphi(1) \in V^\Lambda$. Then for $f \in P_1$ we have

$$\varphi(\partial_0(f)) = \partial_0(f)\varphi(1) = \epsilon(\partial_0(f))\varphi(1) = 0$$

by the exactness of the resolution, hence $\varphi \in H^0(\Lambda, V)$. Conversely, take $\varphi \in H^0(\Lambda, V)$. Then $\ker(\epsilon) = \mathrm{im}(\partial_0) \subseteq \ker(\varphi)$. For $\lambda \in \Lambda$ we have that $\lambda - \epsilon(\lambda)$ lies in the kernel of ϵ, hence

$$\lambda\varphi(1) = (\lambda - \epsilon(\lambda))\varphi(1) + \epsilon(\lambda)\varphi(1) = \varphi(\lambda - \epsilon(\lambda)) + \epsilon(\lambda)\varphi(1) = \epsilon(\lambda)\varphi(1).$$

Therefore $\varphi(1) \in V^\Lambda$. Thus the restriction of the above isomorphism to $H^0(\Lambda, V)$ yields the desired isomorphism. □

2 Regular sequences and Cohen-Macaulay rings

In this section, all rings and algebras (except for Hopf algebras) are associative, commutative, and with unit. If $a_1, \ldots, a_k \in R$ are elements of a ring, we write

$$(a_1, \ldots, a_k)R = \{a_1 f_1 + \cdots + a_k f_k \mid f_1, \ldots, f_k \in R\}$$

for the ideal generated by the a_i. We recall the definition of a regular sequence.

Definition 2.1. *Let R be a ring. A sequence $a_1, \ldots, a_k \in R$ is called R-regular if the following two conditions are satisfied.*

(a) $(a_1, \ldots, a_k)R \neq R$, and

(b) for $1 \leq i \leq k$, multiplication with a_i is injective on $R/(a_1, \ldots, a_{i-1})R$.

The goal of this section is to study the following question: If R is an algebra over a Hopf algebra Λ and $a_1, \ldots, a_k \in R^\Lambda$ are invariants forming an R-regular sequence, under which condition is this sequence also R^Λ-regular? For reasons that will become clear later, we will be especially interested in regular sequences of length 3.

Let R be a ring and $a_1, a_2, a_3 \in R$. We consider the **Koszul complex** $\mathcal{K}(a_1, a_2, a_3; R)$, which in this case is given by

$$0 \longrightarrow R \xrightarrow{\partial_2} R^3 \xrightarrow{\partial_1} R^3 \xrightarrow{\partial_0} R,$$

where the R-homomorphisms ∂_i are as follows: If e_1, e_2, e_3 is the standard basis for R^3, then $\partial_0(e_i) = a_i$. Furthermore, if we label the standard basis vectors of the other R^3 by $e_{i,j}$ $(1 \le i < j \le 3)$, then $\partial_1(e_{i,j}) = a_i e_j - a_j e_i$. Finally, $\partial_2(1) = a_1 e_{2,3} - a_2 e_{1,3} + a_3 e_{1,2}$. It is a general fact that the Koszul complex $\mathcal{K}(a_1, \dots, a_k; R)$ is exact if $a_1, \dots, a_k$ is R-regular (see Eisenbud [4, Corollary 17.5]). Since we need a partial converse for this result, we will give an elementary proof for the case $k = 3$.

Lemma 2.2. *Let R be a ring and $a_1, a_2, a_3 \in R$.*

(a) If a_1, a_2, a_3 is R-regular, then $\mathcal{K}(a_1, a_2, a_3; R)$ is exact.

(b) If the part

$$R^3 \xrightarrow{\partial_1} R^3 \xrightarrow{\partial_0} R$$

of $\mathcal{K}(a_1, a_2, a_3; R)$ is exact, then multiplication with a_3 is injective on $R/(a_1, a_2)R$.

Proof. (a) Since a_1 is not a zero-divisor, ∂_2 in $\mathcal{K}(a_1, a_2, a_3; R)$ is injective. Now suppose that $\partial_1(y_1 e_{2,3} + y_2 e_{1,3} + y_3 e_{1,2}) = 0$. This means that

$$(-y_2 a_3 - y_3 a_2) e_1 + (-y_1 a_3 + y_3 a_1) e_2 + (y_1 a_2 + y_2 a_1) e_3 = 0.$$

From the regularity it follows that $y_1 = z a_1$ with $z \in R$. Now $(z a_2 + y_2) a_1 = 0$ and $(-z a_3 + y_3) a_1 = 0$, hence $y_2 = -z a_2$ and $y_3 = z a_3$. Therefore $y_1 e_{2,3} + y_2 e_{1,3} + y_3 e_{1,2} = \partial_2(z)$. This shows the exactness at the left R^3. To prove the exactness at the right R^3, suppose $\partial_0(x_1 e_1 + x_2 e_2 + x_3 e_3) = 0$, i.e., $x_1 a_1 + x_2 a_2 + x_3 a_3 = 0$. By the regularity, this implies $x_3 \in (a_1, a_2)R$, so $x_3 = y_2 a_1 + y_1 a_2$ with $y_i \in R$. We obtain the equation

$$(x_1 + y_2 a_3) a_1 + (x_2 + y_1 a_3) a_2 = 0,$$

from which $x_2 + y_1 a_3 = y_3 a_1$ with $y_3 \in R$ follows by the regularity. Hence $x_2 = y_3 a_1 - y_1 a_3$. Finally, we obtain

$$(x_1 + y_2 a_3 + y_3 a_2) a_1 = 0,$$

which yields $x_1 = -y_2 a_3 - y_3 a_2$. By the definition of ∂_1, it now follows that

$$x_1 e_1 + x_2 e_2 + x_3 e_3 = \partial_1(y_1 e_{2,3} + y_2 e_{1,3} + y_3 e_{1,2}),$$

which completes the proof of (a).

(b) Suppose that we have $x_3 a_3 \in (a_1, a_2)R$ with $x_3 \in R$. Then $x_1 a_1 + x_2 a_2 + x_3 a_3 = 0$ with $x_i \in R$, hence $\partial_0(x_1 e_1 + x_2 e_2 + x_3 e_3) = 0$. By the exactness, there exist $y_1, y_2, y_3 \in R$ such that

$$x_1 e_1 + x_2 e_2 + x_3 e_3 = \partial_1(y_1 e_{2,3} + y_2 e_{1,3} + y_3 e_{1,2}).$$

Evaluating the e_3-coefficient of this equation yields $x_3 = y_1 a_2 + y_2 a_1$. Therefore $x_3 \in (a_1, a_2)R$, as claimed. □

We return to the question posed after Definition 2.1 and give a cohomological criterion which decides whether an R-regular sequence $a_1, a_2, a_3 \in R^\Lambda$ is also R^Λ-regular. Notice that since multiplication with an invariant is a Λ-endomorphism on R by Proposition 1.3(a), it induces endomorphisms on the cohomology spaces $H^i(\Lambda, R)$.

Theorem 2.3. *Let Λ be a Hopf algebra, R a Λ-algebra, and $a_1, a_2, a_3 \in R^\Lambda$ a R-regular sequence consisting of invariants. Then a_1, a_2, a_3 is R^Λ-regular if and only if the homomorphism*

$$H^1(\Lambda, R) \longrightarrow H^1(\Lambda, R)^3 \tag{2.1}$$

induced by multiplication with the a_i is injective.

Proof. We have $1 \notin (a_1, a_2, a_3)R$ and therefore also $1 \notin (a_1, a_2, a_3)R^\Lambda$. Hence the condition (a) in Definition 2.1 is satisfied. Moreover, multiplication with a_1 is injective on R and hence also on R^Λ. Now suppose that $x_1 a_1 = x_2 a_2$ with $x_1, x_2 \in R^\Lambda$. Then $x_2 = y a_1$ with $y \in R$. We claim that $y \in R^\Lambda$. Indeed, for $\lambda \in \Lambda$ we have

$$a_1(\lambda - \epsilon(\lambda))y = (\lambda - \epsilon(\lambda))(a_1 y) = (\lambda - \epsilon(\lambda))x_2 = 0,$$

hence $(\lambda - \epsilon(\lambda))y = 0$, as claimed. Therefore multiplication with a_2 is injective on $R^\Lambda/(a_1)R^\Lambda$. Thus we have to show that multiplication with a_3 is injective on $R^\Lambda/(a_1, a_2)R^\Lambda$ if and only if (2.1) is injective. By Lemma 2.2 the first condition is equivalent to the exactness of the sequence

$$(R^\Lambda)^3 \longrightarrow (R^\Lambda)^3 \longrightarrow R^\Lambda \tag{2.2}$$

from $\mathcal{K}(a_1, a_2, a_3; R^\Lambda)$. Set $M := \ker(\partial_0) \subseteq R^3$ with ∂_0 from $\mathcal{K}(a_1, a_2, a_3; R)$. Since the ∂_i are Λ-homomorphisms, restriction to the invariants yields an exact sequence

$$0 \longrightarrow M^\Lambda \longrightarrow (R^\Lambda)^3 \longrightarrow R^\Lambda.$$

By Lemma 2.2(a), $M = \operatorname{im}(\partial_1)$, so we obtain the commutative diagram

$$\begin{tikzcd}
& M^\Lambda \arrow[dr] & & \\
(R^\Lambda)^3 \arrow[ur] \arrow[rr] & & (R^\Lambda)^3 \arrow[r] & R^\Lambda.
\end{tikzcd}$$

Thus (2.2) is exact if and only if $(R^\Lambda)^3 \to M^\Lambda$ is surjective. From the short exact sequence

$$0 \longrightarrow R \xrightarrow{\partial_2} R^3 \longrightarrow M \longrightarrow 0$$

we obtain, using Lemma 1.5,

$$0 \longrightarrow R^\Lambda \longrightarrow (R^\Lambda)^3 \longrightarrow M^\Lambda \longrightarrow H^1(\Lambda, R) \longrightarrow H^1(\Lambda, R^3).$$

Therefore $(R^\Lambda)^3 \to M^\Lambda$ is surjective if and only if the map $H^1(\Lambda, R) \to H^1(\Lambda, R^3)$ induced by ∂_2 is injective. But from the definition of ∂_2, this is equivalent to the injectivity of the map (2.1). Summing up, we have proved that a_1, a_2, a_3 is R^Λ-regular if and only if (2.1) is injective. $\square$

Remark 2.4. There is also a version of Theorem 2.3 giving a criterion for the R^Λ-exactness of longer sequences, under a certain condition. It can be found in Kemper [10, Satz 1.9] and states the following: Assume for a non-negative integer r that $H^i(\Lambda, R) = 0$ for all i

with $0 < i < r$. Then an R-regular sequence $a_1, \ldots, a_{r+2} \in R^\Lambda$ is R^Λ-regular if and only if the map

$$H^r(\Lambda, R) \longrightarrow H^r(\Lambda, R)^{r+2}$$

induced by multiplication with the a_i is injective.

Theorem 2.3 is the version for $r = 1$ (in which case the hypothesis is vacuous). ◁

The following example illustrates a situation where the map (2.1) is not injective.

Example 2.5. Let $\{0\} \neq G \leq K^+$ be a subgroup of the additive group of K. We write elements of G as σ_c, $c \in K$. G acts on the polynomial ring $R := K[x_1, x_2, x_3, y_1, y_2, y_3]$ by

$$\sigma_c(x_i) = x_i \quad \text{and} \quad \sigma_c(y_i) = y_i + cx_i.$$

This makes R into an algebra over $\Lambda := KG$. The homomorphism

$$\varphi \colon G \to K^+, \ \sigma_c \mapsto c$$

defines a non-zero element of $H^1(G, K)$. K occurs as the polynomials of degree 0 in R and is hence a direct summand. Therefore φ defines a non-zero element $\alpha \in H^1(G, R)$ via the embedding $H^1(G, K) \hookrightarrow H^1(G, R)$. The sequence $x_1, x_2, x_3 \in R^G$ is clearly R-regular, but we have

$$x_i \varphi(\sigma_c) = cx_i = (\sigma_c - 1) y_i,$$

which is a coboundary. Hence $x_i \alpha = 0$, so the map (2.1) is not injective, and x_1, x_2, x_3 is not R^G-regular. ◁

In fact it is very common (to say the least) that elements in $H^i(\Lambda, R)$ are torsion elements. To illustrate this, consider the case $\Lambda = KG$ with G a finite group, and let the Λ-algebra R be a domain. Then $H^i(G, \mathrm{Quot}(R)) = 0$ for $i > 0$ by (the additive version of) Hilbert's Theorem 90. Thus every element in $H^i(G, R)$ is torsion. However, this consideration does not provide sufficient information to decide if a non-zero element in $H^1(G, R)$ exists whose annihilator in R^G contains an R-regular sequence of length 3.

The main goal of this paper is to study the Cohen-Macaulay property of invariant rings, which we define now.

Definition 2.6. *Let R be an algebra over a field K.*

(a) A sequence $a_1, \ldots, a_k \in R$ is called a **partial system of parameters** *if $(a_1, \ldots, a_k)R \neq R$ and $\mathrm{ht}\,((a_1, \ldots, a_k)R) = k$. Recall that the height of an ideal I is the minimal height of the prime ideals containing I. By Krull's principal ideal theorem (see Eisenbud [4, Theorem 10.2]), we have $\mathrm{ht}\,((a_1, \ldots, a_k)R) \leq k$ for any $a_1, \ldots, a_k \in R$ satisfying $(a_1, \ldots, a_k)R \neq R$.*

(b) R is called **Cohen-Macaulay** *if it is noetherian and every partial system of parameters is an R-regular sequence.*

Remark 2.7. (a) It can be shown that any R-regular sequence is a partial system of parameters (this follows from Eisenbud [4, Theorem 3.1(b)]).

(b) If R has finite Krull dimension n, then n is the maximal length of a partial system of parameters, and a partial system of parameters of this length is called a **system of parameters**.

(c) In Definition 2.6 we gave an alternative definition of Cohen-Macaulayness. The "official" one is as follows: R is called Cohen-Macaulay if for every prime ideal $P \in \mathrm{Spec}(R)$ there exists an R_P-regular sequence in P_P (the maximal ideal of the local ring R_P) of length equal to the Krull dimension $\dim(R_P)$.

(d) If R is a graded algebra and $K = R_0$ is the homogeneous part of degree 0, then R is Cohen-Macaulay if and only if for some (and then for every) homogeneous system of parameters $f_1, \ldots, f_n$, R is a free module over the subalgebra $K[f_1, \ldots, f_n]$. We also remark that for any homogeneous system of parameters R is a finitely generated module over $K[f_1, \ldots, f_n]$ (independently of the Cohen-Macaulay property). This characterization shows what a nice property Cohen-Macaulayness is. In the sequel we shall focus on graded algebras. ◁

Example 2.8. (a) A polynomial ring $K[x_1, \ldots, x_n]$ is Cohen-Macaulay. This can be seen, for example, by using Remark 2.7(d).

(b) If $|G| < \infty$ in Example 2.5, then R^G is not Cohen-Macaulay. Indeed, R is integral over R^G is this case, and we see from Eisenbud [4, Proposition 9.2] that

$$\mathrm{ht}\left((x_1, x_2, x_3)R^G\right) = \mathrm{ht}\left((x_1, x_2, x_3)R\right) = 3.$$

Hence x_1, x_2, x_3 is a partial system of parameters for R^G, but not R^G-regular. ◁

If V is a module over a cocommutative Hopf algebra Λ, then the symmetric algebra $S(V)$ is a Λ-module. (We need cocommutativity so that the ideal factored out from the tensor algebra to form $S(V)$ is a Λ-submodule.) In the case where Λ is the group ring of a linearly reductive group, the celebrated theorem of Hochster and Roberts says the following.

Theorem 2.9 (Hochster and Roberts [8])**.** *Let G be a linearly reductive algebraic group over an algebraically closed field K and V a G-module (i.e., a KG-module such that the action $G \to \mathrm{GL}(V)$ is a morphism of algebraic groups). Then $S(V)^G$ is Cohen-Macaulay.*

The main goal of this paper is to study, in the more general setting of Hopf algebras, to what extent a converse of Theorem 2.9 holds. In order to do so, we must generalize the concept of a linearly reductive group. Here we face the difficulty that linear reductivity is restricted to G-modules, which form a subcategory of all KG-modules. We address this difficulty by allowing subcategories of the module category with certain properties.

Definition 2.10. *Let Λ be a cocommutative Hopf algebra with antipode.*

(a) A full subcategory $\mathcal{C}$ of the module category $\mathcal{MOD}(\Lambda)$ of Λ is called **admissible** *if the following properties hold:*

(i) All objects of $\mathcal{C}$ have finite K-dimension.

(ii) K is an object of $\mathcal{C}$.

(iii) If V is an object of $\mathcal{C}$, then so are all symmetric powers $S^i(V)$ and all submodules $U \leq V$.

(iv) *If U and V are objects of $\mathcal{C}$, then so are $U \oplus V$ and* $\mathrm{Hom}_K(U, V)$.

(b) *If $\mathcal{C}$ is an admissible subcategory of $\mathcal{MOD}(\Lambda)$, then Λ is called* **$\mathcal{C}$-linearly reductive** *if every short exact sequence of modules from $\mathcal{C}$ splits.*

(c) *If $\mathcal{C}$ is an admissible subcategory of $\mathcal{MOD}(\Lambda)$, then Λ is called* **$\mathcal{C}$-geometrically reductive** *if for every module V from $\mathcal{C}$ and for every $0 \neq v \in V^\Lambda$ there exist an $r > 0$ and an $f \in S^r(V^*)^\Lambda$ such that $f(v) \neq 0$.*

Examples of admissible subcategories of $\mathcal{MOD}(\Lambda)$ are the category of all finite dimensional Λ-modules, or the category of all G-modules if $\Lambda = KG$ with G an algebraic group over an algebraically closed field K. In the latter case, Λ is $\mathcal{C}$-linearly or $\mathcal{C}$-geometrically reductive if and only if G is linearly or geometrically reductive, respectively. We will see in Proposition 3.1 that linear reductivity implies geometric reductivity. The main goal of this paper is to prove the following theorem.

Theorem 2.11 (Main Theorem)**.** *Let Λ be a cocommutative Hopf algebra with antipode and $\mathcal{C}$ an admissible subcategory of $\mathcal{MOD}(\Lambda)$ such that Λ is $\mathcal{C}$-geometrically reductive. Then the following two statements are equivalent:*

(a) *Λ is $\mathcal{C}$-linearly reductive.*

(b) *For every object V of $\mathcal{C}$, the invariant ring $S(V)^\Lambda$ is Cohen-Macaulay.*

A corollary of Theorem 2.11 is that if G is a finite group and K a field whose characteristic divides $|G|$, then there exists a finite dimensional KG-module V such that $S(V)^G$ is not Cohen-Macaulay. To interpret Theorem 2.11 for other examples of cocommutative Hopf algebras with antipodes requires an analysis of the different notions of reductivity for these Hopf algebras.

The proof of the implication "(a) $\Rightarrow$ (b)" is a straight forward generalization of Hochster and Roberts's arguments (or, more precisely, the proof due to F. Knop given in Bruns and Herzog [3, Section 6.5]). The proof of the converse implication "(b) $\Rightarrow$ (a)" breaks down into a number of steps, as follows:

(1) If Λ is not $\mathcal{C}$-linearly reductive, we find an object U of $\mathcal{C}$ such that $H^1(\Lambda, U) \neq 0$.

(2) For $\alpha \in H^1(\Lambda, U)$, we construct a module W from $\mathcal{C}$ such that W has a non-zero invariant $0 \neq w \in W^\Lambda$ with $w \otimes \alpha = 0$ in $H^1(\Lambda, W \otimes U)$. In other words, we hire a killer for α.

(3) Then we build the Λ-module

$$V := W \oplus W \oplus W \oplus U.$$

Using Theorem 2.3, we conclude that the three copies w_1, w_2, w_3 of w in V form a sequence in $S(V)^\Lambda$ which is not $S(V)^\Lambda$-regular.

(4) Using the hypothesis that Λ is $\mathcal{C}$-geometrically reductive, we show that w_1, w_2, w_3 is a partial system of parameters for $S(V)^\Lambda$. Thus $S(V)^\Lambda$ is not Cohen-Macaulay.

3 The proof of the Main Theorem

In this section Λ is always a cocommutative Hopf algebra with antipode, and $\mathcal{C}$ is an admissible subcategory of $\mathcal{MOD}(\Lambda)$. We will first prove the implication "(a) $\Rightarrow$ (b)" and then embark on the proof of the converse. First of all, however, we show that linear reductivity implies geometric reductivity.

Proposition 3.1. *If Λ is $\mathcal{C}$-linearly reductive, it is also $\mathcal{C}$-geometrically reductive.*

Proof. Let V be a Λ-module from $\mathcal{C}$ and $0 \neq v \in V^\Lambda$. Using the rule $\epsilon \circ \eta = \epsilon$ (Sweedler [22, Proposition 4.0.1(3)]), it is routine to check that the map $\varphi\colon V^* \to K,\ f \mapsto f(v)$ is a Λ-homomorphism. It follows that if $U \leq V^*$ is an irreducible submodule with $\varphi(U) \neq 0$, then $U \cong K$. By the linear reductivity, V^* is decomposable into irreducible Λ-submodules, and therefore $\varphi(V^*) = \varphi\left((V^*)^\Lambda\right)$. But $v \neq 0$ implies $\varphi \neq 0$, hence there exists $f \in (V^*)^\Lambda$ with $f(v) = \varphi(f) \neq 0$. □

We now prove that for linearly reductive Hopf algebras there exists a so-called Reynolds operator.

Proposition 3.2. *Assume that Λ is $\mathcal{C}$-linearly reductive and that V is a Λ-module from $\mathcal{C}$. Then with $R := S(V)$ there exists a Λ-homomorphism $\pi\colon R \to R^\Lambda$ with $\pi|_{R^\Lambda} = \mathrm{id}$. Any such π is also a homomorphism of R^Λ-modules. It is called a* **Reynolds operator***.*

Proof. For each i the embedding $S^i(V)^\Lambda \hookrightarrow S^i(V)$ splits by the linear reductivity. This yields Λ-homomorphisms π_i which we put together to obtain π. Now fix an $a \in R^\Lambda$ and consider the Λ-endomorphism

$$\varphi\colon R \to R,\ f \mapsto \pi(af).$$

If $U \leq R$ is an irreducible submodule with $\varphi(U) \neq 0$, then $U \cong K$. Take $f \in S^i(V)$ and set $f' := f - \pi(f)$. Since $S^i(V)$ is decomposable into a direct sum of irreducible submodules, we conclude that $\varphi(f') = 0$. Therefore

$$\pi(af) = \varphi(\pi(f) + f') = \varphi(\pi(f)) = \pi(a\pi(f)) = a\pi(f).$$

Now the second claim follows. □

Lemma 3.3. *In the situation of Proposition 3.2, let $I \subseteq R^\Lambda$ be an ideal and let IR be the ideal generated by I in R. Then*

$$IR \cap R^\Lambda = I.$$

Proof. The inclusion $I \subseteq IR \cap R^\Lambda$ is clear. Conversely, take $f = a_1b_1 + \cdots + a_rb_r \in IR \cap R^\Lambda$ with $a_i \in R$ and $b_i \in I$. Applying the Reynolds operator, we obtain

$$f = \pi(f) = \pi(a_1)b_1 + \cdots + \pi(a_r)b_r \in I.$$

□

We can now use Theorem 6.5.2 from Bruns and Herzog [3] to generalize Theorem 2.9.

Theorem 3.4. *Suppose that Λ is $\mathcal{C}$-linearly reductive and V is a Λ-module from $\mathcal{C}$. Then $S(V)^\Lambda$ is Cohen-Macaulay.*

Proof. We first remark that $S(V)^\Lambda$ is a graded K-algebra. Hence Hilbert's classical proof of the finite generation of invariant rings (see, for example, the proof of Theorem 2.1.3 in Sturmfels [21]) is applicable. Thus $S(V)^\Lambda$ is finitely generated and therefore noetherian. Now the claim follows from Bruns and Herzog [3, Theorem 6.5.2], where the essential hypothesis is provided by Lemma 3.3. $\square$

We are now going to take the first step in proving the implication "(b) $\Rightarrow$ (a)" from Theorem 2.11. Fix a Λ-module V from $\mathcal{C}$. A short exact sequence

$$0 \longrightarrow V \longrightarrow W \longrightarrow K \longrightarrow 0 \tag{3.1}$$

with W an object of $\mathcal{C}$ gives rise to the exact sequence

$$0 \longrightarrow V^\Lambda \longrightarrow W^\Lambda \longrightarrow K \xrightarrow{\delta} H^1(\Lambda, V).$$

We see that (3.1) splits if and only if $\delta(1) = 0$. Denote by $H^1_{\mathcal{C}}(\Lambda, V)$ the subset of $H^1(\Lambda, V)$ consisting of all $\delta(1)$ which are obtained from exact sequences (3.1) with W objects of $\mathcal{C}$.

Proposition 3.5. *Λ is $\mathcal{C}$-linear reductive if and only if $H^1_{\mathcal{C}}(\Lambda, V) = 0$ for all Λ-modules V from $\mathcal{C}$.*

Proof. If Λ is $\mathcal{C}$-linear reductive, every short exact sequence of the type (3.1) splits, hence $H^1_{\mathcal{C}}(\Lambda, V) = 0$.

Conversely, assume that $H^1_{\mathcal{C}}(\Lambda, V) = 0$, so that every sequence (3.1) splits. We first claim that an epimorphism $U \to V$ of Λ-modules from $\mathcal{C}$ restricts to an epimorphism $U^\Lambda \to V^\Lambda$. Indeed, take any invariant $0 \neq v \in V^\Lambda$, and let $U' \subseteq U$ be the preimage of Kv. Since Kv is a submodule of V, U' is also a submodule of U. By the hypothesis, the epimorphism $U' \to Kv$ splits. Thus v has a preimage in $(U')^\Lambda \subseteq U^\Lambda$, which proves the claim.

Now let

$$0 \longrightarrow V \longrightarrow W \longrightarrow U \longrightarrow 0 \tag{3.2}$$

be any short exact sequence with objects of $\mathcal{C}$. This leads to an exact sequence

$$0 \longrightarrow \mathrm{Hom}_K(U, V) \longrightarrow \mathrm{Hom}_K(W, V) \longrightarrow \mathrm{Hom}_K(V, V) \longrightarrow 0.$$

We have $\mathrm{id}_V \in \mathrm{Hom}_\Lambda(V, V) = \mathrm{Hom}_K(V, V)^\Lambda$ by Lemma 1.4. By the above, id_V has a preimage $\varphi \in \mathrm{Hom}_K(W, V)^\Lambda = \mathrm{Hom}_\Lambda(W, V)$, again using Lemma 1.4. This φ yields the desired splitting of the sequence (3.2). $\square$

I owe the idea of the current version of the preceding proof to Hanspeter Kraft. We now address the second step in the proof, the construction of a module W with an invariant vector $0 \neq w \in W^\Lambda$ which kills a given $\alpha \in H^1(\Lambda, V)$. The homomorphism $V \to W \otimes V$, $v \mapsto w \otimes v$ induces a map $H^1(\Lambda, V) \to H^1(\Lambda, W \otimes V)$. We denote the image of α under this map by $w \otimes \alpha$.

Proposition 3.6. *Let V be a Λ-module from $\mathcal{C}$ and $\alpha \in H^1_{\mathcal{C}}(\Lambda, V)$. Then there exists a Λ-module W from $\mathcal{C}$ and a non-zero invariant $0 \neq w \in W^\Lambda$ such that $w \otimes \alpha = 0$.*

Proof. We start by considering any Λ-module W with a non-zero invariant $0 \neq w \in W^\Lambda$ and derive conditions on W which are equivalent to $w \otimes \alpha = 0$. In order to find a short exact sequence to which $w \otimes \alpha$ is associated, we have to look at the connection between short exact sequences and first cohomology in more detail. Indeed, α arises from an exact sequence

$$0 \longrightarrow V \longrightarrow \widetilde{V} \xrightarrow{\pi} K \longrightarrow 0$$

with $\widetilde{V}$ a module from $\mathcal{C}$. In explicit terms, this means that α is given by the homomorphism ψ_1 in the commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & P_1 \arrow[r, "\partial_1"] \arrow[d, "\psi_1"] & P_0 \arrow[r, "\partial_0"] \arrow[d, "\psi_0"] & K \arrow[r] \arrow[d, "||"] & 0 \\
0 \arrow[r] & V \arrow[r] & \widetilde{V} \arrow[r, "\pi"] & K \arrow[r] & 0,
\end{tikzcd}$$

where the upper row is a projective resolution and the homomorphisms ψ_i owe their existence to projectivity. We have a Λ-homomorphism $\varphi\colon K \to W,\ c \mapsto cw$. Taking the pullback

$$M := (W \otimes \widetilde{V}) \times_W K := \{(a, c) \in (\widetilde{V} \otimes W) \oplus K \mid (\mathrm{id}_W \otimes \pi)(a) = \varphi(c)\},$$

we obtain a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & W \otimes V \arrow[r] \arrow[d, "||"] & M \arrow[r] \arrow[d] & K \arrow[r] \arrow[d, "\varphi"] & 0 \\
0 \arrow[r] & W \otimes V \arrow[r] & W \otimes \widetilde{V} \arrow[r, "\mathrm{id}_W \otimes \pi"] & W \arrow[r] & 0
\end{tikzcd}$$

with exact rows. We claim that $w \otimes \alpha$ is the element of $H^1(\Lambda, W \otimes V)$ associated to the upper row. Indeed, it is easily checked that the homomorphisms

$$\begin{aligned}
&\psi_0'\colon P_0 \to M,\ x \mapsto (w \otimes \psi_0(x), \partial_0(x)) \quad \text{and} \\
&\psi_1'\colon P_1 \to W \otimes V,\ x \mapsto w \otimes \psi_0(\partial_1(x)) = w \otimes \psi_1(x)
\end{aligned}$$

make the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & P_1 \arrow[r, "\partial_1"] \arrow[d, "\psi_1'"] & P_0 \arrow[r, "\partial_0"] \arrow[d, "\psi_0'"] & K \arrow[r] \arrow[d, "||"] & 0 \\
0 \arrow[r] & W \otimes V \arrow[r] & M \arrow[r] & K \arrow[r] & 0
\end{tikzcd}$$

commute. So we have seen that the sequence $0 \to W \otimes V \to M \to K \to 0$ splits if and only if $w \otimes \alpha = 0$. By the definition of M, the sequence splits if and only if there exists a Λ-homomorphism $K \to W \otimes \widetilde{V}$ such that

$$\begin{tikzcd} & K \arrow[dl] \arrow[d, "\varphi"] \\ W \otimes \widetilde{V} \arrow[r, "\mathrm{id}_W \otimes \pi"] & W \end{tikzcd}$$

commutes. The natural isomorphism $W \otimes \widetilde{V} \xrightarrow{\sim} \mathrm{Hom}_K(\widetilde{V}^*, W)$ and the homomorphism $\mathrm{Hom}_K(\widetilde{V}^*, W) \to W$, $f \mapsto f(\pi)$ make the diagram

$$\begin{tikzcd} W \otimes \widetilde{V} \arrow[r, "\mathrm{id}_W \otimes \pi"] \arrow[dr, "\cong"'] & W \\ & \mathrm{Hom}_K(\widetilde{V}^*, W) \arrow[u] \end{tikzcd}$$

commute. Using this, we conclude that $w \otimes \alpha = 0$ if and only if there exists a Λ-homomorphism $K \to W$ such that

$$\begin{tikzcd} & K \arrow[dl] \arrow[d, "\varphi"] \\ \mathrm{Hom}_K(\widetilde{V}^*, W) \arrow[r] & W \end{tikzcd}$$

commutes. This is equivalent to the existence of $f \in \mathrm{Hom}_K(\widetilde{V}^*, W)^\Lambda = \mathrm{Hom}_\Lambda(\widetilde{V}^*, W)$ (by Lemma 1.4) such that $f(\pi) = w$. Thus W has a $0 \neq w \in W^\Lambda$ with $w \otimes \alpha = 0$ if and only if there exists a Λ-homomorphism $f\colon \widetilde{V}^* \to W$ with $f(\pi) \neq 0$. Hence we can choose $W = \widetilde{V}^*$ (which is an object of $\mathcal{C}$) and $f = \mathrm{id}$. □

Suppose that Λ is not $\mathcal{C}$-linearly reductive. Putting things together as suggested at the end of Section 2, we obtain a Λ-module V from $\mathcal{C}$ and an $S(V)$-regular sequence $w_1, w_2, w_3 \in S(V)^\Lambda$ such that every w_i annihilates a non-zero $\alpha \in H^1(\Lambda, S(V))$, which exists by Proposition 3.5. Therefore w_1, w_2, w_3 is not $S(V)^\Lambda$-regular by Theorem 2.3. The proof of Theorem 2.11 is complete if we can show that w_1, w_2, w_3 is a partial system of parameters for $S(V)^\Lambda$. For this we will need the hypothesis that Λ be $\mathcal{C}$-geometrically reductive.

Lemma 3.7. *Let $\varphi\colon A \to B$ be an epimorphism of Λ-algebras which need not have a unit. Assume that every $a \in A$ lies in a Λ-submodule of A which is an object of $\mathcal{C}$, and that Λ is $\mathcal{C}$-geometrically reductive. Then for every $b \in B^\Lambda$ there exists an $a \in A^\Lambda$ and an $r > 0$ such that $\varphi(a) = b^r$.*

Proof. We follow the proof given by Mumford et al. [14, Lemma A.1.2] of the analogous result for geometrically reductive groups. We can assume that $b \neq 0$. Choose an $a' \in A$ with $\varphi(a') = b$, and a submodule $E \subseteq A$ from $\mathcal{C}$ containing a'. Since b is an invariant, we obtain for $\lambda \in \Lambda$:

$$\lambda a' \in \epsilon(\lambda) a' + (E \cap \ker(\varphi)).$$

Hence $V := Ka' \oplus (E \cap \ker(\varphi)) \subseteq E$ is also a submodule from $\mathcal{C}$. There exists an $f \in V^*$ with $f(a') = 1$ and $f|_{E \cap \ker(\varphi)} = 0$. For $\lambda \in \Lambda$ with $\Delta(\lambda) = \sum_i \mu_i \otimes \nu_i$, $\alpha \in K$ and $v \in E \cap \ker(\varphi)$ we have

$$(\lambda f)(\alpha a' + v) = \sum_i \mu_i \cdot f\left(\eta(\nu_i) \cdot (\alpha a' + v)\right) =$$
$$= \sum_i \epsilon(\mu_i) \cdot f\left(\epsilon(\nu_i)\alpha a'\right) = \epsilon(\lambda)\alpha = (\epsilon(\lambda) \cdot f)(\alpha a' + v),$$

so f lies in $(V^*)^\Lambda$. By the geometric reductivity there exist $r > 0$ and $\widetilde{a} \in S^r(V^{**})^\Lambda$ with $\widetilde{a}(f) = 1$. Using (3), (4), and (6) from Sweedler [22, Proposition 4.0.1], we verify that the canonical embedding $V \to V^{**}$ is a Λ-homomorphism. Since V is finite dimensional, we can therefore assume $\widetilde{a} \in S^r(V)$. The inclusion $V \subseteq A$ induces a Λ-homomorphism ψ: $S^r(V) \to A$. The image of ψ is the direct sum of $K \cdot (a')^r$ with a K-vector space all of whose elements are multiples of elements from $E \cap \ker(\varphi)$. Thus we obtain

$$\varphi(\psi(\widetilde{a})) = b^r,$$

so $a := \psi(\widetilde{a})$ satisfies $\varphi(a) = b^r$. □

From this lemma we derive a "geometric" version of Lemma 3.3, which holds under the hypothesis of geometric reductivity. For the case of a geometrically reductive group, this result is well known (see Newstead [17, Lemma 3.4.2]).

Proposition 3.8. *Assume that Λ is $\mathcal{C}$-geometrically reductive. Let V be a Λ-module from $\mathcal{C}$, $R = S(V)$ the symmetric algebra, and $I \subseteq R^\Lambda$ an ideal. Then*

$$\sqrt{IR} \cap R^\Lambda = \sqrt{I}.$$

Proof. The inclusion $\sqrt{I} \subseteq \sqrt{IR} \cap R^\Lambda$ is clear. For the converse inclusion it suffices to show

$$IR \cap R^\Lambda \subseteq \sqrt{I}.$$

Take $f = \sum_{i=1}^m a_i f_i \in IR \cap R^\Lambda$ with $a_i \in R$, $f_i \in I$. Consider the algebra $A := R[t_1, \ldots, t_m]_+$ of all polynomials in indeterminates $t_1, \ldots, t_m$ and coefficients in R with 0 as constant coefficient. A becomes a Λ-algebra (without unit) with trivial action on the t_i. Every element from A lies in a submodule from $\mathcal{C}$. We have a homomorphism φ: $A \to R$ of Λ-algebras given by $at_i \mapsto af_i$ for $a \in R$. If B is the image of this homomorphism, then $f \in B^\Lambda$. Moreover,

$$\varphi(A^\Lambda) = \varphi(R^\Lambda[t_1, \ldots, t_m]_+) \subseteq I.$$

Now by Lemma 3.7 there exists $r > 0$ such that $f^r \in I$. This completes the proof. □

Remark 3.9. Proposition 3.8 has important geometrical consequences. With the notation of Proposition 3.8, consider the map

$$\pi_\Lambda\colon X := \operatorname{Spec}(R) \to X /\!\!/ \Lambda := \operatorname{Spec}(R^\Lambda)$$

obtained from intersecting a prime ideal from R with R^Λ. This is the **categorical quotient** of the Λ-action on R. Now Proposition 3.8 implies that π_Λ is onto if Λ is $\mathcal{C}$-geometrically reductive. ◁

We deduce the following lemma from Proposition 3.8, which will be used in the proof of Theorem 2.11.

Lemma 3.10. *Suppose that Λ is $\mathcal{C}$-geometrically reductive and V is a Λ-module from $\mathcal{C}$. Then if $v_1, \ldots, v_k \in V^\Lambda \subset S(V)^\Lambda$ are linearly independent over K, they form a partial system of parameters for $S(V)^\Lambda$.*

Proof. We write $R = S(V)$. Clearly $I := (v_1, \ldots, v_k)R^\Lambda \neq R^\Lambda$. We have to show that $\operatorname{ht}(I) \geq k$. The ideals $P_i := (v_1, \ldots, v_i)R \cap R^\Lambda$ $(0 \leq i \leq k)$ are prime ideals in R^Λ forming a strictly increasing sequence. Let $P \subseteq R^\Lambda$ be any prime ideal containing I. Then by Proposition 3.8 we obtain

$$P_k = \sqrt{P_k} = \sqrt{(v_1, \ldots, v_k)R} \cap R^\Lambda = \sqrt{IR} \cap R^\Lambda = \sqrt{I} \subseteq \sqrt{P} = P.$$

Thus $\operatorname{ht}(P) \geq \operatorname{ht}(P_k) \geq k$, which completes the proof. □

Proof of Theorem 2.11. The implication "(a) ⇒ (b)" was shown in Theorem 3.4. Suppose that Λ is not $\mathcal{C}$-linearly reductive. By Theorem 3.5 there exists an object U of $\mathcal{C}$ such that $H^1_{\mathcal{C}}(\Lambda, U) \neq 0$. Choose $0 \neq \alpha \in H^1_{\mathcal{C}}(\Lambda, U)$. By Proposition 3.6 there exists an object W of $\mathcal{C}$ with an invariant $0 \neq w \in W^\Lambda$ such that $w \otimes \alpha = 0$. Form the Λ-module

$$V := U \oplus W \oplus W \oplus W$$

and consider $R = S(V)$. By Lemma 3.10, the three copies w_1, w_2, w_3 of w in V form a partial system of parameters for R^Λ. They also form an R-regular sequence. Since U is a direct summand of R, we can view α as a non-zero element of $H^1(\Lambda, R)$. Applying the map $H^1(\Lambda, R \otimes R) \to H^1(\Lambda, R)$ induced by the multiplication, we find that $w_i\alpha = 0$ for all i. Thus w_1, w_2, w_3 is not R^Λ-regular by Theorem 2.3. Therefore by Definition 2.6(b) R is not Cohen-Macaulay. This completes the proof. □

Remark 3.11. The way in which the hypothesis of geometric reductivity comes into the proof of Theorem 2.11 may seem odd, and possibly unnecessary. However, the example of the additive group G_a (of $\mathbb{C}$, say) shows that this hypothesis is indispensable. Indeed, for every G_a-module V we have that $S(V)^{G_a}$ is Cohen-Macaulay (see Kemper [10, Anmerkung 2.16] or [12, Remark 8] for details). G_a is the prototype of a non-geometrically reductive group. ◁

4 Interpretations of the Main Theorem

In Example 1.2 we saw a few examples of cocommutative Hopf algebras with antipodes. In order to reap some fruits from Theorem 2.11, we have to analyze the reductivity properties for the various instances of Hopf algebras. Thus we can hope to obtain interesting theorems on invariants of groups and of Lie algebras, and possibly other types of objects. Before considering the various cases of Hopf algebras, we proof one general result.

Proposition 4.1. *Let Λ be a finite dimensional Hopf algebra over a field K, and let $\mathcal{C}$ be an admissible subcategory of $\mathcal{MOD}(\Lambda)$. Then Λ is $\mathcal{C}$-geometrically reductive.*

Proof. Let V be an object of $\mathcal{C}$ and take $0 \neq v \in V^\Lambda$. Then there exists an $f \in V^*$ with $f(v) = 1$. By Ferrer Santos [5, Theorem 4.2.1], the finite dimensionality of Λ implies that R is integral over R^Λ. Thus there exist $a_1, \ldots, a_r \in S(V^*)^\Lambda$ such that

$$f^r + a_1 f^{r-1} + \cdots + a_{r-1} f + a_r = 0.$$

Without loss we can assume that $a_i \in S^i(V^*)^\Lambda$. Evaluating the above equation at v yields

$$1 + a_1(v) + \cdots + a_{r-1}(v) + a_r(v) = 0,$$

hence at least one of the $a_i(v)$ must be non-zero. This means that Λ is $\mathcal{C}$-geometrically reductive. □

4.1 Algebraic groups

We consider the case where G is a linear algebraic group over an algebraically closed field K and $\mathcal{C}$ is the subcategory of $\mathcal{MOD}(KG)$ consisting of the G-modules (i.e., the finite dimensional KG-modules V such that the action $G \to \mathrm{GL}(V)$ is given by a morphism of algebraic groups). Theorem 2.11 now yields:

Theorem 4.2 ([12, Theorem 7])**.** *A geometrically reductive algebraic group G is linearly reductive if and only if for all G-modules V the invariant ring $S(V)^G$ is Cohen-Macaulay.*

Remark 4.3. Using the language of schemes, we can also consider a linear algebraic group G over a non-algebraically closed field K. This poses a few difficulties, since the category of G-modules is not a subcategory of the Λ-modules, with Λ the group ring of the group $G(K)$ of K-rational points of G. These difficulties can be circumvented by taking a detour over the algebraic closure of K (see [10, Section 2.2]). The result is that Theorem 4.2 also holds over non-algebraically closed fields. ◁

The following corollary is implicit in [11, Corollary 2.4].

Corollary 4.4. *Let G be a finite group and K a field. Then $S(V)^G$ is Cohen-Macaulay for all finite dimensional KG-modules V if and only if* $\mathrm{char}(K) \nmid |G|$.

Proof. Let $\mathcal{C}$ be the category of all finite dimensional modules over $\Lambda := KG$. By Proposition 4.1, Λ is $\mathcal{C}$-geometrically reductive. Moreover, Λ is $\mathcal{C}$-linearly reductive if and only if $\mathrm{char}(K) \nmid |G|$. Now the corollary follows from Theorem 2.11. □

We consider the two notions of reductivity for linear algebraic groups. Concerning geometric reductivity, it was shown by Haboush [6] that G is geometrically reductive if and only if it is (group theoretically) reductive, i.e., the unipotent radical of G is trivial. Therefore the classical groups are examples of geometrically reductive groups. If the characteristic of K is 0, then G is reductive if and only if it is linearly reductive (see Springer [20, V., Satz 1.1]). Therefore Theorem 4.2 is void in characteristic 0. The situation is remarkably different in positive characteristic. Here the linearly reductive groups were classified by Nagata [15]. The result is that G is linearly reductive if and only if the number of connected components of G is not divisible by $\mathrm{char}(K)$, and the 1-component G^0 is a torus. Thus the two concepts of reductivity fall very much apart in positive characteristic, and Theorem 4.2 is far from void here. For example, we get the result that for the classical groups in positive characteristic there are modules whose invariant rings are not Cohen-Macaulay.

Theorem 4.2 ties in nicely with results of Popov [18] and Nagata [16] to give the following characterization of linearly reductive groups by their invariants.

Corollary 4.5 ([12, Corollary 9]). *Let G be a linear algebraic group over an algebraically closed field K. Then G is linearly reductive if and only if the following two conditions are satisfied:*

(a) If G acts on an affine K-variety X by a morphism $G \times X \to X$, then the invariant ring $K[X]^G$ (with $K[X]$ the coordinate ring) is finitely generated, and

(b) for every G-module V the invariant ring $S(V)^G$ is Cohen-Macaulay.

Proof. Suppose that G is linearly reductive. Then (a) holds by the theorem of Hilbert and Nagata (see [16]), and (b) follows from Theorem 2.9. Conversely, if (a) and (b) are satisfied, then G is reductive by Popov [18]. Therefore Theorem 4.2 applies and yields that G is linearly reductive. □

4.2 Lie algebras and abelian group schemes

Let us call a Lie algebra $\mathfrak{g}$ linearly or geometrically reductive if the universal enveloping algebra $\Lambda = U(\mathfrak{g})$ is $\mathcal{C}$-linearly or $\mathcal{C}$-geometrically reductive, respectively, where $\mathcal{C}$ is the category of all finite dimensional Λ-modules. An analysis of these properties yields:

Proposition 4.6. *Let $\mathfrak{g}$ be a finite dimensional Lie algebra over a field K.*

(a) If the characteristic of K is zero, we have the equivalence

$$\mathfrak{g} \text{ is linearly reductive} \quad \Leftrightarrow \quad \mathfrak{g} \text{ is geometrically reductive} \quad \Leftrightarrow \quad \mathfrak{g} \text{ is semisimple.}$$

(b) If K has positive characteristic, the $\mathfrak{g}$ is geometrically reductive. Moreover, it is linearly reductive if and only if $\mathfrak{g} = 0$.

Proof. We start with characteristic 0. If $\mathfrak{g}$ is semisimple, it is linearly reductive by a theorem of Hermann Weyl (see Bourbaki [2, Chap. I, § 6, Theorem 2]). Linear reductivity implies geometric reductivity by Proposition 3.1. Hence we only have to show that geometric reductivity implies semisimplicity. We will not give the proof here but instead refer the reader to Kemper [10, Proposition 2.8].

Now we consider the case of positive characteristic. If $\mathfrak{g}$ is not the zero-algebra, then by Jacobson [9] there exists a representation which is not completely reducible, hence $\mathfrak{g}$ is not linearly reductive. Now let $\mathfrak{g}$ be an arbitrary finite dimensional Lie algebra in positive characteristic p, and V a finite dimensional $\mathfrak{g}$-module. By the definition of the comultiplication (see Example 1.2(b)), it follows that $\mathfrak{g}$ acts by derivations on $S(V^*)$. Therefore $f^p \in S(V^*)^{\mathfrak{g}}$ for any $f \in S(V^*)$. This implies the geometric reductivity of $\mathfrak{g}$. □

It is quite surprising how different the behavior is whether we are in characteristic 0 or in positive characteristic. In one case, both notions of reductivity coincide, and in the other, they fall as far apart as they possibly can. From Proposition 4.6 we see that Theorem 2.11 does not yield any result for Lie algebras in characteristic 0. For positive characteristic, we obtain the following result.

Theorem 4.7. *Let $\mathfrak{g} \neq 0$ be a finite dimensional Lie algebra over a field of positive characteristic. Then there exists a finite dimensional $\mathfrak{g}$-module V such that $S(V)^{\mathfrak{g}}$ is not Cohen-Macaulay.*

We only report on the reductivity of commutative, cocommutative Hopf algebras with antipodes, i.e., coordinate rings of abelian affine group schemes (see Example 1.2).

Proposition 4.8 ([10, Proposition 2.20])**.** *Let $\Lambda = K[G]$ be the coordinate ring of an affine, smooth, abelian group scheme G of finite type (see Mumford* et al. *[14, Definition 0.2]), and let $\mathcal{C}$ be the category of all finite dimensional Λ-modules.*

(a) Λ is $\mathcal{C}$-linearly reductive if and only if $\dim_K(\Lambda) < \infty$, i.e., G is zero-dimensional.

(b) If $\operatorname{char}(K) = 0$, then Λ is $\mathcal{C}$-geometrically reductive if and only if it is $\mathcal{C}$-linearly reductive.

(c) If K has positive characteristic, then Λ is $\mathcal{C}$-geometrically reductive.

The proof makes essential use of the theorem of Lie-Kolchin. The proof of part (c) is quite similar to the proof of (b) in Proposition 4.6. So again both concepts of reductivity coincide in characteristic 0, and fall apart drastically in positive characteristic. Having made this observation for all classes of cocommutative Hopf algebras with antipode that we considered prompts the following conjecture.

Conjecture 4.9. *Let Λ be a cocommutative Hopf algebra with antipode over a field of characteristic 0 and $\mathcal{C}$ an admissible subcategory of $\mathcal{MOD}(\Lambda)$. Then Λ is $\mathcal{C}$-linearly reductive if and only if it is $\mathcal{C}$-geometrically reductive.*

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