Loci in Quotients by Finite Groups, Pointwise Stabilizers and the Buchsbaum Property

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Abstract

Let $K[V]^G$ be the invariant ring of a finite linear group $G \leq \mathrm{GL}(V)$, and let G_U be the pointwise stabilizer of a subspace $U \leq V$. We prove that the following numbers associated to the invariant ring do not increase if one passes from $K[V]^G$ to $K[V]^{G_U}$: the minimal number of homogeneous generators, the maximal degree of the generators, the number of syzygies and other Betti numbers, the complete intersection defect, the difference between depth and dimension, and the type. From this, theorems of Steinberg, Serre, Nakajima, Kac and Watanabe, and the author follow, which say that if $K[V]^G$ is a polynomial ring, a hypersurface, a complete intersection, or Cohen-Macaulay, then the same is true for $K[V]^{G_U}$. Furthermore, $K[V]^{G_U}$ inherits the Gorenstein property from $K[V]^G$. We give an algorithm which transforms generators of $K[V]^G$ into generators of $K[V]^{G_U}$.

Let $\mathcal{P}$ be one of the properties mentioned above. We consider the locus of $\mathcal{P}$ in $V/\!\!/G := \mathrm{Spec}\left(K[V]^G\right)$ and prove that for $x \in \mathrm{Spec}\left(K[V]\right)$ with image x' in $V/\!\!/G$, the local ring $K[V]^G_{x'}$ has the property $\mathcal{P}$ if and only if $\mathcal{P}$ holds for the invariant ring $K[V]^{G_x}$ of the point stabilizer. Using this, we prove that the non-Cohen-Macaulay locus in $V/\!\!/G$ is either empty, or it has dimension at least one and codimension at least 3. From this we deduce that $K[V]^G$ is Buchsbaum if and only if it is Cohen-Macaulay. This proves a conjecture of Campbell et al..

Introduction

In a beautiful survey article, Stanley [29] described a number of structural properties which an invariant ring $K[V]^G$, or any other graded algebra R, may have. These properties are ordered hierarchically as follows:

$$\begin{aligned} R \text{ polynomial ring} \implies R \text{ hypersurface} \implies R \text{ complete intersection} \\ \implies R \text{ Gorenstein} \implies R \text{ Cohen-Macaulay.} \end{aligned} \tag{1}$$

Like Stanley, we restrict our attention to the case where G is a finite group acting linearly on a finite dimensional vector space V over a field K, but there is no restriction on K. We write $K[V]$ for the symmetric algebra of the dual V^*, and $K[V]^G$ for the invariant ring. If K is infinite, $K[V]^G$ can be viewed as the ring of polynomial functions on V which are constant on all G-orbits. For a subgroup $H \leq G$, no general rules hold about the behavior of the above properties when one passes from the G-invariants to the H-invariants. However, the situation is much better if H arises as the pointwise stabilizer G_U of a linear subspace $U \leq V$. In fact, this has been a recurrent subject of research, and far-reaching results are known. For example, if $K[V]^G$ is a polynomial ring, then by

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a result due to Serre (see Bourbaki [3, Chapter V, § 6, Ex. 8] or Nakajima [25, Lemma 1.4]) the same is true for $K[V]^{G_U}$. In characteristic zero, this is the result of Steinberg [30], which says that the pointwise stabilizers of a reflection group are again reflection groups. Moreover, if $K[V]^G$ is a hypersurface (resp. a complete intersection), then so is $K[V]^{G_U}$ by Nakajima [28, Lemma 2.5] (resp. Kac and Watanabe [15]). Finally, the author [18] proved that the Cohen-Macaulay property passes from $K[V]^G$ to $K[V]^{G_U}$. It follows from the results of this paper that the same is true for the Gorenstein property (see Theorem B). Results of this type can be very useful for proving that a certain property does not hold for $K[V]^G$. For example, Nakajima [25] and Kemper and Malle [21] used Serre's result to show that many invariant rings of modular reflection groups are not polynomial rings. See also Example 2.18. In the context of reductive groups in characteristic 0, the philosophy that the invariant ring $K[L]^{G_x}$ of the slice representation L for a point x with closed orbit becomes simpler than $K[V]^G$ has also been known for a while and was used by Kac et al. [16] and Kac [14], to name just a few examples.

Apart from structural properties, there are also interesting numerical invariants associated to a Noetherian graded algebra $R = \bigoplus_{d=0}^{\infty} R_d$ over a field $K = R_0$. The first is the minimal number k of homogeneous generators of R. R is a polynomial algebra if and only if $k = \dim(R)$, so we can interpret the non-negative integer

$$\operatorname{pdef}(R) := (\text{minimal number of homogeneous generators of } R) - \dim(R)$$

as the polynomial defect of R. A further invariant which has been the subject of intensive research is $\beta(R)$, the maximal degree of an element in a minimal homogeneous generating set, which is independent of the choice of the generators. In other words, $\beta(R)$ is the minimal number d such that R can be generated by homogeneous elements of degree at most d.

If R is not a polynomial ring, there exist relations between minimal homogeneous generators of R. Let $b_1(R)$ be the minimal number of relations which generate the ideal of relations. This number does not depend on the choice of the minimal generators. It follows from Krull's principal ideal theorem that $b_1(R)$ is bigger than or equal to $\operatorname{pdef}(R)$, and equality holds if and only if R is a complete intersection. Thus the non-negative integer

$$\operatorname{cidef}(R) := b_1(R) - \operatorname{pdef}(R)$$

can be interpreted as the complete intersection defect. Giving homogeneous generators for R amounts to giving an epimorphism $\varphi\colon S \to R$ from a graded polynomial algebra S onto R, and giving generators for the ideal of relations is the same as giving an epimorphism from a free S-module F_1 onto $\ker(\varphi)$. The syzygies of the second kind are generators of the kernel of this latter epimorphism. Continuing this way, we obtain a free resolution

$$\cdots \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow R \longrightarrow 0 \tag{2}$$

of R as an S-module. If in each step the generators of the previous kernel are chosen homogeneous and minimal, the ranks of the F_i only depend on R. We write

$$b_i(R) := \operatorname{rank}(F_i) \tag{3}$$

and call the $b_i(R)$ the Betti numbers. They encode the homological complexity of R. R is called Cohen-Macaulay if its depth (the maximal length of a homogeneous regular sequence) equals its dimension. So let us call the non-negative integer

$$\operatorname{cmdef}(R) := \dim(R) - \operatorname{depth}(R)$$

the Cohen-Macaulay defect. By the Auslander-Buchsbaum formula (see Benson [2, Theorem 4.4.4]), $\operatorname{cmdef}(R) + \operatorname{pdef}(R) = \operatorname{hdim}_S(R) := \max\{i \mid b_i(R) \neq 0\}$, so we see how the Cohen-Macaulay defect is encoded by the Betti numbers, and what it says about the complexity of R. Finally R is Gorenstein if it is Cohen-Macaulay of type one. This brings a further invariant, the type $r(R)$, into the play. We recall the definition in Section 2.3.

In this paper we prove that all these numerical invariants become smaller or stay the same if one passes from $K[V]^G$ to $K[V]^{G_U}$. More precisely, we prove:

Theorem A. *For a Noetherian graded algebra R over a field $K = R_0$, let $f(R)$ be one of the functions $\operatorname{pdef}(R)$, $\beta(R)$, $\operatorname{cidef}(R)$, $\operatorname{cmdef}(R)$, $r(R)$, or $b_i(R)$ introduced above. Let $G \leq \operatorname{GL}(V)$ be a finite linear group on a finite dimensional vector space V over K, and let $U \leq V$ be a linear subspace. Then for the pointwise stabilizer*

$$G_U := \{\sigma \in G \mid \sigma(x) = x \text{ for all } x \in U\}$$

the following statements hold:

(a) $f\left(K[V]^{G_U}\right) \leq f\left(K[V]^G\right)$.

(b) If U has a G_U-stable complement in V, then $f\left(K[V]^{G_U}\right) = f\left(K[V/U]^{G_U}\right)$.

(c) With $m := \operatorname{pdef}\left(K[V]^G\right) - \operatorname{pdef}\left(K[V]^{G_U}\right)$, *the stronger inequality*

$$\sum_{j=0}^{m} \binom{m}{j} b_{i-j}\left(K[V]^{G_U}\right) \leq b_i\left(K[V]^G\right)$$

holds. (Here we set $b_i(R) := 0$ for $i < 0$.)

Note that by Maschke's theorem the hypothesis of (b) is always satisfied if $\operatorname{char}(K) \nmid |G_U|$.

As a side product from the proof we obtain an algorithm which converts generating invariants for $K[V]^G$ into generators of $K[V]^{G_U}$ (see Theorem 2.7). Theorem A has the following immediate consequence.

Theorem B. *Let $\mathcal{P}$ be one of the properties occurring in* (1)*. Then in the situation of Theorem A we have:*

(a) $\mathcal{P}\left(K[V]^G\right)$ *implies* $\mathcal{P}\left(K[V]^{G_U}\right)$.

(b) If U has a G_U-stable complement in V, then $\mathcal{P}\left(K[V]^{G_U}\right)$ *and* $\mathcal{P}\left(K[V/U]^{G_U}\right)$ *are equivalent.*

As mentioned above, Theorem B is already known for all properties except the Gorenstein property.

The properties $\mathcal{P}$ occurring in (1), with "polynomial ring" replaced by "regular", are also applicable to local rings. This raises the question of the locus of $\mathcal{P}$: for which $x \in V/\!\!/G := \operatorname{Spec}\left(K[V]^G\right)$ does $\mathcal{P}\left(K[V]^G_x\right)$ hold? We give a description of the loci in terms of the invariant rings of point stabilizers. Let $\mathfrak{q} \in \operatorname{Spec}(K[V])$ be a prime ideal. (We abbreviate elements from $\operatorname{Spec}(K[V])$ or $\operatorname{Spec}\left(K[V]^G\right)$ by letters x, y or $\mathfrak{q}, \mathfrak{p}$, depending on whether we view them as points or as prime ideals.) Then we write

$$G_{\mathfrak{q}} := \{\sigma \in G \mid \sigma(f) - f \in \mathfrak{q} \text{ for all } f \in K[V]\} \tag{4}$$

for the **point stabilizer** (or isotropy subgroup) of $\mathfrak{q}$. If $\mathfrak{q}$ is given by a point $x \in V$, $G_{\mathfrak{q}}$ is simply the set of elements in G which fix x. If $\mathfrak{q}$ is the ideal in $K[V]$ generated by the linear forms vanishing on a subspace $U \leq V$, then $G_U = G_{\mathfrak{q}}$ with G_U defined as in Theorem A. Conversely, for any $\mathfrak{q} \in \operatorname{Spec}(K[V])$, let $U := (V^* \cap \mathfrak{q})^{\perp} \leq V$ be the subspace annihilated by all linear forms lying in $\mathfrak{q}$. Then it is easy to see that $G_{\mathfrak{q}} = G_U$. Thus the subgroups of G occurring as $G_{\mathfrak{q}}$'s or G_U's are the same, and we might as well have stated Theorems A and B for point stabilizers $G_{\mathfrak{q}}$ or even for pointwise stabilizers of arbitrary subsets $S \subseteq \operatorname{Spec}(K[V])$. We write

$$\pi_G\colon \operatorname{Spec}(K[V]) \to V/\!\!/G, \ \mathfrak{q} \mapsto K[V]^G \cap \mathfrak{q}$$

for the **categorical quotient**. Note that since $K[V]$ is integral over $K[V]^G$, π_G is surjective (see Eisenbud [7, Proposition 4.15]). We will prove:

Theorem C. *Assume that K is a perfect field, and let $\mathcal{P}$ be one of the properties occurring in (1). Then for $x \in \operatorname{Spec}(K[V])$ and $x' := \pi_G(x)$ we have*

$$\mathcal{P} \text{ holds for } K[V]^G_{x'} \text{ if and only if } \mathcal{P} \text{ holds for } K[V]^{G_x}.$$

The hypothesis that K be perfect is a very mild one, and in fact it is only necessary for the properties of being a polynomial ring or a hypersurface. Since there are algorithms to decide which of the properties in (1) hold for the invariant ring of a finite group (see Kemper [17]), Theorem C enables us to explicitly determine the loci of these properties.

Using Theorem C, we prove that the non-Cohen-Macaulay locus (which is a closed subvariety in $V/\!\!/G$) is either empty, or has dimension at least one and codimension at least 3. Stronger bounds hold for the dimension of the locus where the Cohen-Macaulay defect is greater than some given number (see Theorem 3.1). This result already appeared in Kemper [18, Korollar 5.15].

A property that was not mentioned in the hierarchy (1) is the Buchsbaum property, which is a weakening of the Cohen-Macaulay property. We recall the definition in Section 3.2. It was proved independently by Nakajima [27] and Campbell et al. [6] that if G is a p-group, then $K[V]^G$ is Buchsbaum if and only if it is Cohen-Macaulay. Later Kemper [20, Theorem 1.7] proved the equivalence for a larger class of groups, which includes all groups with $V^G \neq 0$ and all groups of order not divisible by $\operatorname{char}(K)^2$. Campbell et al. [6, Conjecture 27] conjectured that the equivalence holds for all finite linear groups G. Using the result on the dimension of the non-Cohen-Macaulay locus, we prove this conjecture (see Theorem 3.4).

The plan of the paper is as follows. In Section 1 we prove that for $x \in V$ the completions of the local rings $K[V]^G_{\pi_G(x)}$ and $K[V]^{G_x}_{\pi_{G_x}(x)}$ are isomorphic. For K algebraically closed, this is a consequence of Luna's slice theorem, but we also give an elementary proof. Everything follows from this isomorphism. We formulate general conditions on a property $\mathcal{P}$ such that theorems on $\mathcal{P}$ which are of the same type as Theorems B and C can be deduced from this isomorphism. In the second section we show how the properties and numerical invariants that we are interested in can be expressed as properties (applicable to local and graded rings) for which the conditions given in Section 1 hold. Together with the results from Section 1, this yields proofs of Theorems A–C. Finally, Section 3 is devoted to the study of the non-Cohen-Macaulay locus and the Buchsbaum property.

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1 Invariants of pointwise stabilizers

In this section we set up the general framework to prove theorems of the type of Theorems A–C.

1.1 An isomorphism

We consider the following situation: Let X be an affine scheme of finite type over a field K and let G be a finite group of K-automorphisms of X. We write $K[X]$ for the coordinate ring of X and $K[X]^G$ for the invariant ring. For $x \in X$ an ideal in $K[X]$, we define the point stabilizer G_x as in (4). We denote the local ring of X at $x \in X$ by $K[X]_x$, and its completion with respect to the maximal ideal by $\widehat{K[X]_x}$.

Proposition 1.1. *In the above situation, let $x \in X$ be a K-geometric point, and write $x' := \pi_G(x)$ and $x'' := \pi_{G_x}(x)$. Then the inclusion $K[X]^G \subseteq K[X]^{G_x}$ induces an isomorphism*

$$\widehat{K[X]^G_{x'}} \xrightarrow{\sim} \widehat{K[X]^{G_x}_{x''}}.$$

Proof. We give two proofs, one of which relies on Luna's slice theorem [23] and works for K algebraically closed and x smooth, and another which is elementary and works in the general situation.

So first assume that K is algebraically closed and x is a smooth point. From the finiteness of G it follows that the G-orbit $G(x)$ of x is closed and separable (see Bardsley and Richardson [1] for the appropriate definitions). Clearly there exists a smooth, G_x-stable affine open subvariety $Y \subseteq X$ containing x, and then Y and $G(x)$ are transversal at x. Thus by Bardsley and Richardson [1, Proposition 7.3] there exists an affine open neighborhood $S \subseteq Y$ of x which is an étale slice at x. In particular, the natural morphism $S/\!\!/G_x \to X/\!\!/G$ is étale. Therefore the induced homomorphism $\widehat{K[X]^G_{x'}} \to \widehat{K[S]^{G_x}_{x''}}$ is an isomorphism. But $K[S]^{G_x}_{x''} \cong K[X]^{G_x}_{x''}$ since S is open in X, and the result follows.

Now we turn to the general proof. The inclusion $K[X]^G \subseteq K[X]^{G_x}$ induces a monomorphism $K[X]^G_{x'} \hookrightarrow K[X]^{G_x}_{x''}$. We wish to show that (a) and (b) from Lemma 1.2 hold for $R = K[X]^G_{x'}$ and $S = K[X]^{G_x}_{x''}$. Being a K-geometric point, x is given as the kernel $\mathfrak{m}$ of a homomorphism $K[X] \to K$ of K-algebras, and x' and x'' are the kernels $\mathfrak{p}$ and $\mathfrak{q}$ of the compositions $K[X]^G \hookrightarrow K[X] \to K$ and $K[X]^{G_x} \hookrightarrow K[X] \to K$, respectively. Since the compositions are onto, we have $K[X]^G/\mathfrak{p} \cong K \cong K[X]^{G_x}/\mathfrak{q}$, which implies (b) of Lemma 1.2.

By the prime avoidance lemma (see Eisenbud [7, Lemma 3.3]), there exists an $f \in \mathfrak{m}$ such that $f \notin \sigma(\mathfrak{m})$ for all $\sigma \in G \setminus G_x$. Then $g := \prod_{\tau \in G_x} \tau(f)$ lies in $\mathfrak{q}$, but $g \notin \sigma(\mathfrak{m})$ for $\sigma \in G \setminus G_x$. Let $J \subseteq \mathfrak{q}$ be a maximal subideal generated by elements not lying in any $\sigma(\mathfrak{m})$ for $\sigma \in G \setminus G_x$. Then

$$\mathfrak{q} \subseteq J \cup \bigcup_{\sigma \in G \setminus G_x} \left(\sigma(\mathfrak{m}) \cap K[X]^{G_x}\right).$$

By the above $\mathfrak{q} \subseteq \sigma(\mathfrak{m}) \cap K[X]^{G_x}$ is impossible for $\sigma \in G \setminus G_x$, hence we conclude by applying prime avoidance again that $\mathfrak{q} = J$. Therefore $\mathfrak{q}$ is generated by elements $f_1, \ldots, f_n \in \mathfrak{q}$ such that $f_i \notin \sigma(\mathfrak{m})$ for $\sigma \in G \setminus G_x$. Choose representatives $\sigma_1, \ldots, \sigma_r$ for the left cosets of G_x in G, excluding G_x itself. Then $g_i := \prod_{j=1}^r \sigma_j(f_i)$ lies in $K[X]^{G_x}$ but not in $\mathfrak{m}$, so g_i is a unit in the localization $K[X]^{G_x}_{\mathfrak{q}} = K[X]^{G_x}_{x''}$. It follows that the maximal ideal $\mathfrak{q}_{\mathfrak{q}}$ of $K[X]^{G_x}_{\mathfrak{q}}$ is generated by the $f_i \cdot g_i$. But $f_i \cdot g_i \in K[X]^G \cap \mathfrak{q} = \mathfrak{p}$, which proves (a) of Lemma 1.2. This completes the proof. □

The following lemma was used in the above proof.

Lemma 1.2. *Let $\varphi\colon R \hookrightarrow S$ be a monomorphism of Noetherian local rings with maximal ideals $\mathfrak{m}$ and $\mathfrak{n}$, respectively, such that*

(a) $\varphi(\mathfrak{m})S = \mathfrak{n}$, and

(b) φ induces an isomorphism $\bar{\varphi}\colon R/\mathfrak{m} \to S/\mathfrak{n}$.

Then the homomorphism $\hat{\varphi}\colon \hat{R} \to \hat{S}$ induced by φ is an isomorphism.

Proof. By Eisenbud [7, Theorem 7.2(a)], assumption (a) implies that the natural map $\hat{R} \otimes_R S \to \hat{S}$ is an isomorphism. Composing $\mathrm{id}_{\hat{R}} \otimes \varphi\colon \hat{R} \to \hat{R} \otimes_R S$ with this isomorphism yields $\hat{\varphi}$. But by the flatness of completion (see Eisenbud [7, Theorem 7.2(b)]), $\mathrm{id}_{\hat{R}} \otimes \varphi$ is injective, hence so is $\hat{\varphi}$.

To show the surjectivity of $\hat{\varphi}$, take $\hat{g} \in \hat{S}$ arbitrary. So $\hat{g}$ is represented by a sequence $(g_i)_{i \in \mathbb{Z}_{>0}}$ with $g_i \in S$ such that $g_j - g_i \in \mathfrak{n}^i$ for $j \geq i$. We will construct f_i recursively such that $f_j - f_i \in \mathfrak{m}^i$ for $j \geq i$ and $\varphi(f_i) - g_i \in \mathfrak{n}^i$. Then the f_i define an $\hat{f} \in \hat{R}$ with $\hat{\varphi}(\hat{f}) = \hat{g}$. Suppose f_{i-1} has been constructed for some $i > 0$ (formally set $f_0 := 0$). Choose generators $x_1, \dots, x_k$ of $\mathfrak{m}$, then by the assumption (a) we have $\mathfrak{n} = (\varphi(x_1), \dots, \varphi(x_k))\, S$. Therefore $\varphi(f_{i-1}) - g_i \in \mathfrak{n}^{i-1}$ implies

$$\varphi(f_{i-1}) - g_i = \sum_m h_m \varphi(m)$$

with $h_m \in S$, where the sum ranges over the monomials m of degree $i-1$ in the x_j. By (b) there exist $r_m \in R$ such that $\varphi(r_m) - h_m \in \mathfrak{n}$. Set $f_i := f_{i-1} - \sum_m r_m m$. Then $f_i - f_{i-1} \in \mathfrak{m}^{i-1}$ and

$$\varphi(f_i) - g_i = \sum_m (h_m - \varphi(r_m))\, \varphi(m) \in \mathfrak{n}^i.$$

These are the desired properties of f_i, hence the surjectivity of $\hat{\varphi}$ is proved. $\square$

Remark. We adopt the notation of Proposition 1.1.

(a) In the case that $X/\!\!/G$ is irreducible, $X/\!\!/G_x$ is reduced and x' is a normal point, Proposition 1.1 also follows from Grothendieck [10, Exposé I, Théorème 10.11].

(b) Let W be a finite dimensional KG-module and $M := K[X] \otimes_K W$. Then M^G is the module of equivariants. It is not clear whether $\widehat{M^G_{x'}}$ and $\widehat{M^{G_x}_{x''}}$ are isomorphic. $\triangleleft$

We specialize our situation by taking X to be a finite dimensional vector space V over K, and $G \leq \mathrm{GL}(V)$ a finite linear group on V. $K[V]^G$ is a graded algebra with unique maximal homogeneous ideal $K[V]^G_+ := \pi_G(0)$ consisting of the invariants vanishing at $0 \in V$.

Proposition 1.3. *Let $G \leq \mathrm{GL}(V)$ be a finite linear group acting on a finite dimensional vector space V over a field K. Then for every point $x \in V$ we have an isomorphism*

$$\widehat{K[V]^G_{\pi_G(x)}} \cong \widehat{K[V]^{G_x}_{\pi_{G_x}(0)}}.$$

Proof. We define an automorphism φ of $K[V]$ (as a K-algebra) by $\varphi(f)\colon v \mapsto f(v+x)$ for $v \in V$. Then φ commutes with the G_x-action and maps the ideal $\mathfrak{p} \subset K[V]$ of polynomials vanishing at x to the ideal $\mathfrak{q}$ of polynomials vanishing at 0. This yields a G_x-equivariant isomorphism $K[V]_\mathfrak{p} \xrightarrow{\sim} K[V]_\mathfrak{q}$, hence

$$(K[V]_\mathfrak{p})^{G_x} \cong (K[V]_\mathfrak{q})^{G_x}.$$

Clearly $(K[V]^{G_x})_{K[V]^{G_x} \cap \mathfrak{p}} \subseteq (K[V]_\mathfrak{p})^{G_x}$, and $f \in K[V] \setminus \mathfrak{p}$ implies $\prod_{\sigma \in G_x} \sigma(f) \in K[V]^{G_x} \setminus \mathfrak{p}$, which yields the reverse inclusion. Correspondingly, $(K[V]^{G_x})_{K[V]^{G_x} \cap \mathfrak{q}} = (K[V]_\mathfrak{q})^{G_x}$, and hence $K[V]^{G_x}_{\pi_{G_x}(x)} \cong K[V]^{G_x}_{\pi_{G_x}(0)}$. Combining this with Proposition 1.1 yields the result. $\square$

1.2 Local and geometric properties

We make the following definitions.

Definition 1.4. *(a) We call a graded ring $R = \oplus_{d=0}^{\infty} R_d$* **local** *if R_0 is a field. In this case $R_+ := \oplus_{d>0} R_d$ is the unique homogeneous maximal ideal of R.*

(b) Suppose that $\mathcal{P}$ is a property applicable to Noetherian local rings. Then for a Noetherian ring R we write

$$\mathrm{Loc}_R(\mathcal{P}) := \{x \in \mathrm{Spec}(R) \mid \mathcal{P}(R_x) \text{ holds}\}$$

for the **locus** *of $\mathcal{P}$*

(c) A property $\mathcal{P}$ applicable to Noetherian local rings and Noetherian local graded rings is called **local** *if the following four conditions hold:*

(L1) For a Noetherian local ring R, $\mathcal{P}(R)$ holds if and only if $\mathcal{P}(\widehat{R})$ holds.

(L2) For a Noetherian local graded ring R, $\mathcal{P}(R)$ holds if and only if $R_+ \in \operatorname{Loc}_R(\mathcal{P})$.

(L3) If $\mathcal{P}(R)$ holds for a Noetherian local graded domain R, then $\operatorname{Loc}_R(\mathcal{P})$ contains all R_0-geometric points in $\operatorname{Spec}(R)$.

(L4) Let R be a Noetherian local graded ring and $S = R_0[x_1, \ldots, x_n]$ a graded polynomial ring with all x_i homogeneous of degree 1. Then $\mathcal{P}(S \otimes_{R_0} R)$ and $\mathcal{P}(R)$ are equivalent.

(d) Let R be a Noetherian ring. A subset $Y \subseteq X := \operatorname{Spec}(R)$ is called **constructible** *if it can be written as a finite union of subsets which are locally closed (i.e., intersections of open and closed sets) in X (see Hartshorne [11, Chapter II, Exercise 3.18]). If Y is constructible, R is an algebra over a field K and L is an extension field, we write $Y_L := \operatorname{Spec}(L) \times_{\operatorname{Spec}(K)} Y \subseteq \operatorname{Spec}(L \otimes_K R)$. Thus Y_L is formed by substituting every ideal $I \subseteq R$ involved in the definition of Y by the ideal in $L \otimes_K R$ generated by I.*

(e) A property $\mathcal{P}$ applicable to Noetherian local rings and Noetherian local graded rings is called **geometric** *(over a field K) if the conditions (L1) and (L2) hold, and for every Noetherian local graded domain R (with $R_0 = K$) the following two conditions hold:*

(G1) $\operatorname{Loc}_R(\mathcal{P})$ is constructible.

(G2) If $\bar{K}$ is an algebraic closure of R_0, then $\operatorname{Loc}_{R\otimes_K \bar{K}}(\mathcal{P}) = \operatorname{Loc}_R(\mathcal{P})_{\bar{K}}$.

Example 1.5. As we will see in Section 2, the properties occurring in (1) are all local and geometric (at least over a perfect field). ◁

The following theorem is a simple consequence of Proposition 1.3.

Theorem 1.6. *Let $G \leq \operatorname{GL}(V)$ be a finite linear group acting on a finite dimensional vector space V over a field K, and let $\mathcal{P}$ be a local property applicable to Noetherian local rings and Noetherian local graded rings. Then we have for every subspace $U \leq V$:*

(a) $\mathcal{P}(K[V]^G)$ implies $\mathcal{P}(K[V]^{G_U})$.

(b) If U has a G_U-stable complement in V, then $\mathcal{P}\left(K[V]^{G_U}\right)$ and $\mathcal{P}\left(K[V/U]^{G_U}\right)$ are equivalent.

Proof. Let U be spanned by the vectors $x_1, \ldots, x_m \in V$. Then $G_U = G_{x_1} \cap \cdots \cap G_{x_m}$. Thus using induction on m reduces the proof of (a) to the case $m = 1$, i.e., $G_U = G_x$. But by (L3) from Definition 1.4, $\mathcal{P}(K[V]^G)$ implies $\mathcal{P}(K[V]^G_{\pi_G(x)})$, which by (L1) is equivalent to $\mathcal{P}(\widehat{K[V]^G_{\pi_G(x)}})$. By Proposition 1.3, this implies $\mathcal{P}(\widehat{K[V]^{G_x}_{\pi_{G_x}(0)}})$, which by (L1) and (L2) is equivalent to $\mathcal{P}(K[V]^{G_x})$. This completes the proof of (a).

If U has a G_U-stable complement in V, then $V \cong V/U \oplus U$ (as modules over the group ring KG_U) and hence $K[V] \cong K[V/U] \otimes_K K[U]$. This implies $K[V]^{G_U} \cong K[V/U]^{G_U} \otimes_K K[U]$, and so the equivalence of $\mathcal{P}\left(K[V]^{G_U}\right)$ and $\mathcal{P}\left(K[V/U]^{G_U}\right)$ follows from (L4). □

Our next goal is to obtain a description of loci in invariant rings. The following result gives this description for the K-geometric points.

Proposition 1.7. *Let $\mathcal{P}$ be a property applicable to Noetherian local rings and Noetherian local graded rings, which satisfies the conditions (L1) and (L2) from Definition 1.4. Then in the situation of Proposition 1.3 we have for $x \in V$:*

$$\pi_G(x) \in \operatorname{Loc}_{K[V]^G}(\mathcal{P}) \quad \textit{if and only if} \quad \mathcal{P}(K[V]^{G_x}) \textit{ holds.}$$

Proof. This is an immediate consequence of Proposition 1.3. □

The next technical lemma is used to generalize Proposition 1.7 to non-geometric points in $V/\!\!/G$.

Lemma 1.8. *Let R be a finitely generated algebra over a field K and $Y, Z \subseteq X := \operatorname{Spec}(R)$ two constructible sets. Then the following statements are equivalent:*

(a) $Y = Z$.

(b) $\operatorname{Spec}_{\max}(R) \cap Y = \operatorname{Spec}_{\max}(R) \cap Z$, *where* $\operatorname{Spec}_{\max}(R)$ *is the set of maximal ideals in* R.

(c) $Y_{\bar{K}} = Z_{\bar{K}}$, *where* $\bar{K}$ *is an algebraic closure of* K.

Proof. We can write

$$Y = \bigcup_{i=1}^{n} \left(\mathcal{V}_X(I_i) \setminus \mathcal{V}_X(J_i)\right)$$

with $I_i, J_i \subseteq R$ ideals. Take any $\mathfrak{p} \in X$. By renumbering the I_i and J_i, we can assume that there exists $k \in \{0, \ldots, n\}$ such that $\mathfrak{p} \notin \bigcup_{i=1}^{k} \mathcal{V}_X(J_i)$ but $\mathfrak{p} \in \mathcal{V}_X(J_i)$ for $i > k$. Thus

$$J := \bigcap_{i=1}^{k} J_i \not\subseteq \mathfrak{p}, \quad \text{and} \quad J_i \subseteq \mathfrak{p} \quad \text{for} \quad i > k.$$

With $I := \bigcap_{i=1}^{k} I_i$ we have $\mathfrak{p} \in Y$ if and only if $I \subseteq \mathfrak{p}$.

We first claim that $\mathfrak{p}$ lies in Y if and only if $\mathfrak{p}$ is the intersection of maximal ideals in Y. From this the equivalence of (a) and (b) follows. So assume that $\mathfrak{p} \in Y$, and let $\mathcal{M} \subseteq \operatorname{Spec}_{\max}(R)$ be the set of maximal ideals containing $\mathfrak{p}$ but not J, and $\mathcal{M}' \subseteq \operatorname{Spec}_{\max}(R)$ the set of maximal ideals containing $\mathfrak{p}$ and J. Since R is a Jacobson ring (see Eisenbud [7, Theorem 4.19]), we have

$$\mathfrak{p} = \bigcap_{\mathfrak{q} \in \mathcal{M}} \mathfrak{q} \cap \bigcap_{\mathfrak{q} \in \mathcal{M}'} \mathfrak{q} = \bigcap_{\mathfrak{q} \in \mathcal{M}} \mathfrak{q} \cap \sqrt{\mathfrak{p} + J}.$$

Since $\sqrt{\mathfrak{p} + J} \not\subseteq \mathfrak{p}$, it follows that $\bigcap_{\mathfrak{q} \in \mathcal{M}} \mathfrak{q} = \mathfrak{p}$. Since $I \subseteq \mathfrak{p}$, it follows that $I \subseteq \mathfrak{q}$ for $\mathfrak{q} \in \mathcal{M}$. Therefore $\mathcal{M} \subseteq Y$. Conversely, assume $\mathfrak{p} = \bigcap_{\mathfrak{q} \in \mathcal{M}} \mathfrak{q}$ with $\mathcal{M} \subseteq \operatorname{Spec}_{\max}(R) \cap Y$. Then for every $\mathfrak{q} \in \mathcal{M}$ and every $i > k$ we have $J_i \subseteq \mathfrak{q}$, and therefore $I \subseteq \mathfrak{q}$ (otherwise, $\mathfrak{q} \notin Y$). This implies $\mathfrak{p} \in Y$.

Now let $\mathfrak{p}$ be maximal. Then we claim that $\mathfrak{p}$ lies in Y if and only if there exists a $\mathfrak{q} \in Y_{\bar{K}}$ with $\mathfrak{p} = R \cap \mathfrak{q}$. Since the equivalence of (a) and (b) is already proved, this implies the equivalence of (a) and (c). So assume $\mathfrak{p} \in Y$ and choose a $\mathfrak{q} \in \operatorname{Spec}(\bar{K} \otimes_K R)$ with $\mathfrak{p} \subseteq \mathfrak{q}$. Then $I \subseteq \mathfrak{q}$ and therefore $I\bar{R} \subseteq \mathfrak{q}$, where we write $\bar{R} := \bar{K} \otimes_K R$. By the maximality of $\mathfrak{p}$ we get $R \cap \mathfrak{q} = \mathfrak{p}$, and therefore $J \not\subseteq \mathfrak{q}$. This implies $J\bar{R} \not\subseteq \mathfrak{q}$, hence $\mathfrak{q} \in Y_{\bar{K}}$. Conversely, assume that $\mathfrak{p} = R \cap \mathfrak{q}$ with $\mathfrak{q} \in Y_{\bar{K}}$. Then $J_i \subseteq \mathfrak{q}$ for $i > k$ and therefore $I\bar{R} \subseteq \mathfrak{q}$ (otherwise $\mathfrak{q} \notin Y_{\bar{K}}$). This implies $I \subseteq \mathfrak{p}$, i.e., $\mathfrak{p} \in Y$. This completes the proof. □

Theorem 1.9. *Let $G \leq \operatorname{GL}(V)$ be a finite linear group acting on a finite dimensional vector space V over a field K, and let $\mathcal{P}$ be a geometric property applicable to Noetherian local rings and Noetherian local graded rings. Then for $x \in \operatorname{Spec}(K[V])$ we have*

$$\pi_G(x) \in \operatorname{Loc}_{K[V]^G}(\mathcal{P}) \quad \textit{if and only if} \quad \mathcal{P}(K[V]^{G_x}) \textit{ holds.}$$

Here G_x is defined by Equation (4).

Proof. Set $X := \operatorname{Spec}(K[V])$. We have to show the equality of the sets

$$Y := \pi_G^{-1}\left(\operatorname{Loc}_{K[V]^G}(\mathcal{P})\right) \quad \text{and} \quad Z := \left\{x \in X \mid \mathcal{P}(K[V]^{G_x}) \text{ holds}\right\}.$$

Y is constructible by Condition (G1) from Definition 1.4. Let $\mathcal{M}$ be the set of all subgroups $H \leq G$ such that $\mathcal{P}(K[V]^H)$ holds, and for $H \leq G$ set $X^H := \{x \in X \mid H \leq G_x\}$. X^H is closed and thus

$$Z = \bigcup_{H \in \mathcal{M}} \left(X^H \setminus \bigcup_{H \lneqq H' \leq G} X^{H'}\right)$$

is constructible. By Lemma 1.8 we have to show that an $x \in \mathrm{Spec}_{\max}(\bar{K} \otimes_K K[V])$ lies in $Y_{\bar{K}}$ if and only if it lies in $Z_{\bar{K}}$. We write $\bar{V} := \bar{K} \otimes_K V$. Then for $H \leq G$ we have $\bar{K} \otimes_K K[V]^H \cong \bar{K}[\bar{V}]^H$ and therefore

$$H \in \mathcal{M} \iff K[V]^H_+ \in \mathrm{Loc}_{K[V]^H}(\mathcal{P}) \iff \bar{K}[\bar{V}]^H_+ \in \mathrm{Loc}_{K[V]^H}(\mathcal{P})_{\bar{K}}$$
$$\iff \bar{K}[\bar{V}]^H_+ \in \mathrm{Loc}_{\bar{K}[\bar{V}]^H}(\mathcal{P}) \iff \mathcal{P}(\bar{K}[\bar{V}]^H) \text{ holds,}$$

where we used (L2) and (G2). Hence

$$Z_{\bar{K}} = \bigcup_{H \in \mathcal{M}} \Big(X^H_{\bar{K}} \setminus \bigcup_{H \lneqq H' \leq G} X^{H'}_{\bar{K}} \Big) = \big\{ x \in X_{\bar{K}} \mid \mathcal{P}(\bar{K}[\bar{V}]^{G_x}) \text{ holds} \big\}.$$

On the other hand,

$$Y_{\bar{K}} = (\bar{K} \otimes \pi_G)^{-1} \left(\mathrm{Loc}_{K[V]^G}(\mathcal{P})_{\bar{K}} \right) = (\bar{K} \otimes \pi_G)^{-1} \left(\mathrm{Loc}_{\bar{K}[\bar{V}]^G}(\mathcal{P}) \right),$$

where we used (G2). Thus we have to show that for $x \in \mathrm{Spec}_{\max}(\bar{K} \otimes_K K[V])$ the property $\mathcal{P}\left(\bar{K}[\bar{V}]^{G_x}\right)$ holds if and only if $(\bar{K} \otimes \pi_G)(x)$ lies in $\mathrm{Loc}_{\bar{K}[\bar{V}]^G}(\mathcal{P})$. But since $\bar{K}$ is algebraically closed, such an x is a $\bar{K}$-geometric point, and therefore the equivalence holds by Proposition 1.7. $\square$

Example 1.10. As an application, we get that for $x \in \mathrm{Spec}(K[V])$, $\pi_G(x)$ lies in the non-singular locus of $V/\!\!/G$ if and only if G_x has polynomial invariants.

Likewise, we obtain from Hochster and Eagon [13] that if the characteristic of K does not divide the group order $|G_x|$, then $\pi_G(x)$ lies in the Cohen-Macaulay locus of $V/\!\!/G$. This was also proved by Broer [4]. $\triangleleft$

Corollary 1.11. *In the situation of Theorem 1.9, let $\mathcal{P}$ be local and geometric over K. Then $\mathrm{Loc}_{K[V]^G}(\mathcal{P})$ is open in $\mathrm{Spec}(K[V]^G)$.*

Proof. We write $X := \mathrm{Spec}(K[V])$ and set

$$\mathcal{M} := \{H \leq G \mid G_{V^H} = H\} \quad \text{and} \quad \mathcal{F} := \left\{H \in \mathcal{M} \mid \mathcal{P}\left(K[V]^H\right) \text{ does not hold}\right\}.$$

We claim that

$$\pi_G^{-1}\left(\mathrm{Loc}_{K[V]^G}(\mathcal{P})\right) = X \setminus \bigcup_{H \in \mathcal{F}} X^H.$$

This implies the corollary, since π_G is a finite morphism and hence closed (see Hartshorne [11, Chapter II, Exercise 3.5]), and π_G is surjective. To prove the claim, take $x \in X$ with $\pi_G(x) \in \mathrm{Loc}_{K[V]^G}(\mathcal{P})$. By Theorem 1.9, $\mathcal{P}\left(K[V]^{G_x}\right)$ holds. Let $H \in \mathcal{M}$ be a subgroup with $x \in X^H$, so $H \leq G_x$. Thus $H \leq (G_x)_{V^H} \leq G_{V^H} = H$ and therefore $H = (G_x)_{V^H}$. Now by Theorem 1.6, $\mathcal{P}\left(K[V]^H\right)$ follows and hence $H \notin \mathcal{F}$. We conclude that $x \notin X^H$ for all $H \in \mathcal{F}$. This yields the first inclusion.

To prove the reverse inclusion, take $x \in X \setminus \bigcup_{H \in \mathcal{F}} X^H$. Thus $G_x \notin \mathcal{F}$. But $G_x = G_U$ for some subspace $U \leq V$ by the remark before Theorem C. Therefore $G_x \in \mathcal{M}$, so $\mathcal{P}\left(K[V]^{G_x}\right)$ must hold. By Theorem 1.9, $\pi_G(x) \in \mathrm{Loc}_{K[V]^G}(\mathcal{P})$ follows. $\square$

2 Properties of invariant rings

With a few possible gaps, it can be picked out from various places in the literature that the properties occurring in (1) are local and geometric. In this section we will give a uniform proof for this fact, which also shows that the properties "$f(R) \leq m$" for f one of the functions mentioned in Theorem A (except $\beta(R)$) are local and geometric. This is possible since all these functions can be expressed in terms of the embedding dimension and the Betti numbers.

2.1 The embedding dimension

Let R be a Noetherian local (graded) ring with maximal (homogeneous) ideal $\mathfrak{m}$. Then we define the **embedding dimension** of R as

$$\operatorname{edim}(R) := \dim_{R/\mathfrak{m}} \left(\mathfrak{m}/\mathfrak{m}^2\right).$$

Notice that $\operatorname{edim}(R) \geq \dim(R)$ for a Noetherian local ring R. It follows from the next proposition (which is folklore) that the same inequality holds for Noetherian local graded rings.

Proposition 2.1. *Let R be a Noetherian local graded ring. Then $\operatorname{edim}(R)$ is the minimal number k such that R can be generated by k elements as an algebra over R_0. Moreover, there exist $\operatorname{edim}(R)$ homogeneous generators of R.*

Proof. First suppose $f_1, \ldots, f_k$ generate R over $K := R_0$. The f_i still generate R if we subtract the components of degree 0, hence we may assume that $f_i \in \mathfrak{m}$. But then it is clear that the $f_i + \mathfrak{m}^2$ generate $\mathfrak{m}/\mathfrak{m}^2$ as a vector space over K, so $k \geq \operatorname{edim}(R)$.

Conversely, choose homogeneous $f_1, \ldots, f_k \in \mathfrak{m}$ such that the $f_i + \mathfrak{m}^2$ form a K-basis of (the graded vector space) $\mathfrak{m}/\mathfrak{m}^2$, so $k = \operatorname{edim}(R)$. Let $f \in R$ be homogeneous of positive degree. Then

$$f = \sum_{i=1}^{k} \alpha_i f_i + \sum_{j=1}^{r} g_j h_j$$

with $\alpha_i \in K$ and $g_j, h_j \in \mathfrak{m}$. We can assume that the g_j are homogeneous with $\deg(g_j h_j) = \deg(f)$. Therefore $\deg(g_j), \deg(h_j) < \deg(f)$, and by induction on $\deg(f)$ we conclude that $g_j, h_j \in K[f_1, \ldots, f_k]$. Thus $f \in K[f_1, \ldots, f_k]$, which proves that $f_1, \ldots, f_k$ generate R. □

Similarly, if R is a Noetherian local ring, then $\operatorname{edim}(R)$ is the minimal dimension of a regular local ring S which has R as an epimorphic image, provided such an S exists (see Bruns and Herzog [5, p. 72]).

Lemma 2.2. *If R is a Noetherian local ring and $\widehat{R}$ its completion, then*

$$\operatorname{edim}(\widehat{R}) = \operatorname{edim}(R).$$

Proof. This is clear since the associated graded rings of R and $\widehat{R}$ coincide (see Eisenbud [7, Theorem 7.1(b)]). □

Lemma 2.3. *Let R be a Noetherian local graded ring with maximal homogeneous ideal $\mathfrak{m} = R_+$. Then*

$$\operatorname{edim}(R_\mathfrak{m}) = \operatorname{edim}(R).$$

Proof. The maximal ideal of $R_\mathfrak{m}$ is $\mathfrak{m}_\mathfrak{m}$. Consider the natural homomorphism φ: $\mathfrak{m} \to \mathfrak{m}_\mathfrak{m}/\mathfrak{m}_\mathfrak{m}^2$. Clearly $\mathfrak{m}^2 \subseteq \ker(\varphi)$. Conversely, take $f \in \ker(\varphi)$. Then there exists $g \in R \setminus \mathfrak{m}$ such that $gf \in \mathfrak{m}^2$. Since $\mathfrak{m}$ is a maximal ideal, there exists $h \in R$ such that $hg - 1 \in \mathfrak{m}$. We have $hgf \in \mathfrak{m}^2$, hence

$$f = (1 - hg)f + hgf \in \mathfrak{m}^2.$$

Therefore $\ker(\varphi) = \mathfrak{m}^2$.

To prove that φ is surjective, take $f/g \in \mathfrak{m}_\mathfrak{m}$ with $f \in \mathfrak{m}$ and $g \in R \setminus \mathfrak{m}$. Again we have $h \in R$ with $hg - 1 \in \mathfrak{m}$, so

$$\frac{hf}{1} - \frac{f}{g} = \frac{hg-1}{1} \cdot \frac{f}{g} \in \mathfrak{m}_\mathfrak{m}^2.$$

Thus $f/g + \mathfrak{m}_\mathfrak{m}^2 = \varphi(hf)$. It follows that $\mathfrak{m}/\mathfrak{m}^2$ and $\mathfrak{m}_\mathfrak{m}/\mathfrak{m}_\mathfrak{m}^2$ are isomorphic as $R/\mathfrak{m}$-modules. But $R/\mathfrak{m} \cong R_\mathfrak{m}/\mathfrak{m}_\mathfrak{m}$ since $\mathfrak{m}$ is maximal. From this the lemma follows. □

Lemma 2.4. *Let R be a Noetherian local graded ring and $\mathfrak{p} \in \operatorname{Spec}(R)$. Then we have:*

(a) $\operatorname{edim}(R_{\mathfrak{p}}) \leq \operatorname{edim}(R)$.

(b) If R is a domain, then $\operatorname{edim}(R_{\mathfrak{p}}) - \dim(R_{\mathfrak{p}}) \leq \operatorname{edim}(R) - \dim(R)$.

Proof. By Proposition 2.1, R can be generated by $k := \operatorname{edim}(R)$ elements. Thus there exists a polynomial ring S over R_0 in k indeterminates and an epimorphism $\pi\colon S \to R$. With $\mathfrak{q} := \pi^{-1}(\mathfrak{p})$, this yields an epimorphism $S_{\mathfrak{q}} \to R_{\mathfrak{p}}$. This epimorphism induces an isomorphism $S_{\mathfrak{q}}/\mathfrak{q}_{\mathfrak{q}} \xrightarrow{\sim} R_{\mathfrak{p}}/\mathfrak{p}_{\mathfrak{p}}$ and an epimorphism $\mathfrak{q}_{\mathfrak{q}}/\mathfrak{q}_{\mathfrak{q}}^2 \to \mathfrak{p}_{\mathfrak{p}}/\mathfrak{p}_{\mathfrak{p}}^2$. Therefore $\operatorname{edim}(R_{\mathfrak{p}}) \leq \operatorname{edim}(S_{\mathfrak{q}})$. Since $S_{\mathfrak{q}}$ is regular (see Bruns and Herzog [5, Theorem 2.2.13 and Corollary 2.2.9]), we obtain

$$\begin{aligned}\operatorname{edim}(R_{\mathfrak{p}}) - \dim(R_{\mathfrak{p}}) \leq \dim(S_{\mathfrak{q}}) - \dim(R_{\mathfrak{p}}) &= \operatorname{ht}(\mathfrak{q}) - \operatorname{ht}(\mathfrak{p}) = \\ &= k - \dim(S/\mathfrak{q}) - \operatorname{ht}(\mathfrak{p}) = k - (\dim(R/\mathfrak{p}) + \operatorname{ht}(\mathfrak{p})).\end{aligned}$$

This yields the inequality (a). Since $\dim(R/\mathfrak{p}) + \operatorname{ht}(\mathfrak{p}) = \dim(R)$ if R is a domain (see Eisenbud [7, Corollary 13.4]), (b) also follows. □

Lemma 2.5. *Let R be a Noetherian local graded ring and $S = R_0[x_1, \ldots, x_n]$ a graded polynomial ring with all x_i homogeneous of degree 1. Then*

$$\operatorname{edim}(S \otimes_{R_0} R) = \operatorname{edim}(R) + n.$$

Proof. We have

$$(S \otimes R)_+ = \bigoplus_{i+j>0} S_i \otimes R_j = S_0 \otimes R_+ \oplus S_+ \otimes R$$

and

$$(S \otimes R)_+^2 = S_0 \otimes R_+^2 \oplus S_1 \otimes R_+ \oplus S_{>1} \otimes R,$$

since $S_{>1} = S_+^2$. Therefore the map $S_+ \oplus R_+ \to (S\otimes R)_+/(S\otimes R)_+^2$, $(s, r) \mapsto (s\otimes 1 + 1\otimes r) + (S\otimes R)_+^2$ is surjective and has kernel $S_+^2 \oplus R_+^2$. This yields the lemma. □

Proposition 2.6. *For $m \geq 0$, define the property $\mathcal{P}_m$ by saying that $\mathcal{P}_m$ holds for a Noetherian local (graded) ring R if* $\operatorname{edim}(R) - \dim(R) \leq m$. *Then $\mathcal{P}_m$ is local and geometric over every perfect field.*

Proof. For a local ring R we have $\dim(\widehat{R}) = \dim(R)$ (see Eisenbud [7, Corollary 10.12]), which together with Lemma 2.2 yields Condition (L1) from Definition 1.4. For a Noetherian local graded ring R with maximal homogeneous ideal $\mathfrak{m}$ we have $\dim(R) = \operatorname{ht}(\mathfrak{m}) = \dim(R_{\mathfrak{m}})$ (see Eisenbud [7, Corollary 13.7]), which together with Lemma 2.3 yields (L2). (L3) follows from Lemma 2.4(b). Moreover, if S is a polynomial ring over a field K, then Noether's normalization lemma yields $\dim(S \otimes_K R) = \dim(S) + \dim(R)$ for a finitely generated K-algebra R. Together with Lemma 2.5 this yields (L4). Thus $\mathcal{P}_m$ is local.

Now let R be a Noetherian local graded domain with $K := R_0$ a perfect field, and take $\mathfrak{p} \in \operatorname{Spec}(R)$. There exists a polynomial ring $S := K[x_1, \ldots, x_k]$ and an ideal $I = (f_1, \ldots, f_r) \subseteq S$ such that $R \cong S/I$. Let $\mathfrak{J} := (\partial f_i/\partial x_j)$ be the Jacobian matrix. Set $L := R_{\mathfrak{p}}/\mathfrak{p}_{\mathfrak{p}}$. From the proof of Theorem 16.19 in Eisenbud [7], we see that

$$\dim_L \left(\mathfrak{p}_{\mathfrak{p}}/\mathfrak{p}_{\mathfrak{p}}^2\right) = k - \operatorname{rank}(\overline{\mathfrak{J}}) - \dim(R/\mathfrak{p}),$$

where $\overline{\mathfrak{J}} \in L^{m\times r}$ denotes $\mathfrak{J}$ reduced modulo $\mathfrak{p}$. This yields

$$\operatorname{edim}(R_{\mathfrak{p}}) - \dim(R_{\mathfrak{p}}) = k - \dim(R/\mathfrak{p}) - \operatorname{ht}(\mathfrak{p}) - \operatorname{rank}(\overline{\mathfrak{J}}) = k - \dim(R) - \operatorname{rank}(\overline{\mathfrak{J}}).$$

Therefore $\mathcal{P}_m(R_{\mathfrak{p}})$ is equivalent to $\operatorname{rank}(\overline{\mathfrak{J}}) \geq k - \dim(R) - m =: s$. But this is equivalent to the condition that the ideal generated by the $s \times s$ minors of $\mathfrak{J}$ is not contained in $\mathfrak{p}$. Hence $\operatorname{Loc}_R(\mathcal{P}_m)$ is open, which proves (G1). (G2) follows from the observation that $\bar{K} \otimes R$ has the presentation $\bar{K} \otimes R = \bar{K}[x_1, \ldots, x_k]/(f_1, \ldots, f_r)$, so $\operatorname{Loc}_{\bar{K}\otimes_K R}(\mathcal{P}_m)$ is again determined by the $s \times s$ minors of $\mathfrak{J}$. □

Thus Theorem 1.9 is applicable to the properties $\mathcal{P}_m$, and from Theorem 1.6 we obtain the result that the invariant ring of a pointwise stabilizer G_U is generated by at most as many invariants as $K[V]^G$, which was already stated in Theorem A in the introduction. This result raises the question whether and how generators of $K[V]^{G_U}$ can be constructed from generators of $K[V]^G$. Since all our proofs are explicit and constructive, this can indeed be done.

So suppose that $f_1, \ldots, f_k$ generate $K[V]^G$ as a K-algebra. Pick a vector x from a basis of U and let $\mathfrak{p} \subseteq K[V]^G$ be the ideal of invariants vanishing at x. Then the images of the $f_i^{(0)} := f_i - f_i(x)$ generate $\mathfrak{p}/\mathfrak{p}^2$ as a vector space over K. Let $\widehat{\mathfrak{p}}$ be the image of $\mathfrak{p}$ in the completion $\widehat{K[V]^G_{\mathfrak{p}}}$. The images of the $f_i^{(0)}$ generate $\widehat{\mathfrak{p}}/\widehat{\mathfrak{p}}^2$. If $\mathfrak{q} \subseteq K[V]^{G_x}$ is the ideal of G_x-invariants vanishing at x and $\widehat{\mathfrak{q}}$ its image in $\widehat{K[V]^{G_x}_{\mathfrak{q}}}$, then by Proposition 1.1 the images of the $f_i^{(0)}$ also generate $\widehat{\mathfrak{q}}/\widehat{\mathfrak{q}}^2$. Now we apply the homomorphism φ: $K[V] \to K[V]$ of K-algebras given by $\varphi(f)(v) = f(v+x)$ to the $f_i^{(0)}$ and write $\widetilde{f}_i := \varphi(f_i^{(0)})$. By the proof of Proposition 1.3, φ induces an isomorphism $\widehat{K[V]^{G_x}_{\mathfrak{q}}} \xrightarrow{\sim} \widehat{K[V]^{G_x}_{\pi_{G_x}(0)}}$. Therefore the images of the $\widetilde{f}_i$ generate $\widehat{\mathfrak{m}}/\widehat{\mathfrak{m}}^2$, where $\widehat{\mathfrak{m}}$ is the image of $\mathfrak{m} := K[V]^{G_x}_+$. By Lemmas 2.2 and 2.3 they also generate $\mathfrak{m}/\mathfrak{m}^2$ (as a K-vector space). Now a homogeneous generating system of $\mathfrak{m}/\mathfrak{m}^2$ of at most k elements can be chosen, and by the proof of Proposition 2.1 this system will also generate $K[V]^{G_x}$. If we allow for an increase in the number of generators, we can also decompose the $\widetilde{f}_i$ into their homogeneous components (which are all members in $\mathfrak{m}$). These components also generate $\mathfrak{m}/\mathfrak{m}^2$ as a K-vector space, and therefore $K[V]^{G_x}$ as a K-algebra. We have proved:

Theorem 2.7. *Let $G \leq \mathrm{GL}(V)$ be a finite linear group acting on a finite dimensional vector space V over a field K, and let $x \in V$ be a point. Apply the endomorphism φ of $K[V]$ given by $\varphi(f)(v) = f(v+x)$ to generators $f_1, \ldots, f_k \in K[V]^G$ of the invariant ring. Then the homogeneous components of the resulting polynomials generate the invariant ring $K[V]^{G_x}$.*

To obtain generators of the invariant ring of a pointwise stabilizer G_U, we can iterate the process in Theorem 2.7 over a basis $x_1, \ldots, x_r$ of U. Although the polynomials $\varphi(f_i)$ are usually not homogeneous, their maximal degree is bounded above by the maximal degree of the f_i. Thus for the maximal degree in a system of homogeneous generators we have the relation

$$\beta\left(K[V]^{G_U}\right) \leq \beta\left(K[V]^G\right),$$

which was stated in Theorem A.

Example 2.8. Let $G = S_n$ be the symmetric group acting on basis vectors $e_1, \ldots, e_n$ of V, and choose $x := e_n$, so $G_x = S_{n-1}$. $K[V]^G$ is generated by the elementary symmetric polynomials $f_{n,i}(x_1, \ldots, x_n)$ ($i = 1, \ldots, n$), where $x_1, \ldots, x_n \in V^*$ is a dual basis. An easy calculation shows that applying φ from Theorem 2.7 yields

$$\varphi(f_{n,i}) := f_{n,i}(x_1, \ldots, x_{n-1}, x_n + 1) = f_{n-1,i} + (x_n + 1) f_{n-1,i-1}$$

with $f_{n-1,0} := 1$. Thus $\varphi(f_{n,i})$ has homogeneous components $f_{n-1,i-1}$ and $f_{n-1,i} + x_n f_{n-1,i-1}$. From these, the non-redundant generators $f_{n-1,i-1}$ ($1 < i \leq n$) and $f_{n-1,1} + x_n$ can be chosen. Subtracting $f_{n-1,1}$ from $f_{n-1,1} + x_n$, we obtain the expected generators x_n and $f_{n-1,i}$ ($i = 1, \ldots, n-1$) of $K[V]^{S_{n-1}}$.

If we consider $G = A_n$ the alternating group and assume $\mathrm{char}(K) \neq 2$, then $K[V]^G$ has the additional generator $g_n := \prod_{1 \leq i < j \leq n} (x_i - x_j)$. Applying φ yields

$$\varphi(g_n) = \prod_{1 \leq i < j \leq n-1} (x_i - x_j) \cdot \prod_{1 \leq i \leq n-1} (x_i - x_n - 1) = g_{n-1} \cdot f$$

with $f \in K[V]^{S_{n-1}}$. Since the degree 0 component of f is $(-1)^{n-1}$, g_{n-1} occurs as a homogeneous component of $(-1)^{n-1} \varphi(g_n)$. Therefore we obtain the expected generators of $K[V]^{G_x} = K[V]^{A_{n-1}}$. $\triangleleft$

A Noetherian local graded ring is called a **hypersurface** if it is a quotient of a polynomial ring over a field by a principal ideal. The next lemma is folklore.

Lemma 2.9. *Let R be a Noetherian local graded domain. Then*

(a) R is (isomorphic to) a polynomial ring over a field if and only if $\mathcal{P}_0$ from Proposition 2.6 holds for R.

(b) R is a hypersurface if and only if $\mathcal{P}_1(R)$ holds.

Proof. First notice that $K := R_0$ is the largest field contained in R.

(a) If R is a polynomial ring, then $R = K[x_1, \dots, x_k]$ with $k = \dim(R)$, so $k = \operatorname{edim}(R)$ follows by Proposition 2.1. Conversely, if $\mathcal{P}_0(R)$ holds, then by Proposition 2.1 we have $R \cong K[x_1, \dots, x_k]/I$ with $k = \dim(R)$. Therefore $I = 0$ and R is polynomial.

(b) If R is a hypersurface, then $R = K[x_1, \dots, x_k]/(f)$, so $\operatorname{edim}(R) \leq k$ and $\dim(R) \geq k - 1$ by Krull's principal ideal theorem (see Eisenbud [7, Theorem 10.2]). This implies $\mathcal{P}_1(R)$. Conversely, suppose $\operatorname{edim}(R) - \dim(R) \leq 1$. By Proposition 2.1 we have $R \cong K[x_1, \dots, x_k]/I$ with $k = \operatorname{edim}(R)$. Hence $\operatorname{ht}(I) \leq 1$. Since I is a prime ideal, every non-zero polynomial in I has a prime factor f lying in I. But then (f) is a non-zero prime ideal contained in I and therefore $I = (f)$. □

Thus Theorem 1.9 can be used to compute the non-singular locus and the "hypersurface locus" of $V /\!\!/ G$, and Theorem 1.6 yields that if $K[V]^G$ is a polynomial ring or a hypersurface, then the same holds for every $K[V]^{G_U}$ with $U \leq V$, as stated in Theorem B.

2.2 Betti numbers

For a Noetherian local (graded) ring R with maximal (homogeneous) ideal $\mathfrak{m}$, choose $x_1, \dots, x_k \in \mathfrak{m}$ whose images are a basis of $\mathfrak{m}/\mathfrak{m}^2$ as a vector space over $K := R/\mathfrak{m}$. Thus $k = \operatorname{edim}(R)$, and by Proposition 2.1 and Nakayama's lemma (see Eisenbud [7, Corollary 4.8]), the x_i generate $\mathfrak{m}$. Let $\mathcal{K}_\bullet(x_1, \dots, x_k)$ be the Koszul complex of $x_1, \dots, x_k$ with coefficients in R, and $H_\bullet(x_1, \dots, x_k) := H_\bullet(\mathcal{K}_\bullet(x_1, \dots, x_k))$ its homology (see Bruns and Herzog [5, Section 1.6]). By Bruns and Herzog [5, Proposition 1.6.5(b)], $\mathfrak{m}$ annihilates $H_\bullet(x_1, \dots, x_k)$. This makes $H_\bullet(x_1, \dots, x_k)$ into a vector space over K, and we define

$$b_i(R) := \dim_K (H_i(x_1, \dots, x_k))$$

for $i \in \mathbb{Z}$. This does not depend on the choice of the generators $x_1, \dots, x_k$ (see Bruns and Herzog [5, p. 52]). We have $b_i(R) = 0$ for $i < 0$ and $b_0(R) = 1$. Let us call the $b_i(R)$ the **Betti numbers** of R. The idea behind this term will become clear by Proposition 2.12.

Lemma 2.10. *Let R be a Noetherian local ring and $\widehat{R}$ its completion. Then for all $i \in \mathbb{Z}$ we have*

$$b_i(\widehat{R}) = b_i(R).$$

Proof. Let $x_1, \dots, x_k$ be minimal generators of the maximal ideal $\mathfrak{m} \subset R$. Then the images $\widehat{x_1}, \dots, \widehat{x_k}$ minimally generate the maximal ideal $\widehat{\mathfrak{m}} \subset \widehat{R}$, and the flatness of completion implies that $H_\bullet(\widehat{x_1}, \dots, \widehat{x_k}) \cong \widehat{R} \otimes_R H_\bullet(x_1, \dots, x_k)$ by Bruns and Herzog [5, Proposition 1.6.7(b)]. The lemma follows. □

Lemma 2.11. *Let R be a Noetherian local graded ring with maximal homogeneous ideal $\mathfrak{m}$. Then for all $i \in \mathbb{Z}$ we have*

$$b_i(R_\mathfrak{m}) = b_i(R).$$

Proof. If $x_1, \dots, x_k$ are minimal generators of $\mathfrak{m}$, then the $x_i/1$ minimally generate the maximal ideal $\mathfrak{m}_\mathfrak{m} \subset R_\mathfrak{m}$. The flatness of localization and Proposition 1.6.7(b) of Bruns and Herzog [5] now imply $H_\bullet(x_1/1, \dots, x_k/1) \cong R_\mathfrak{m} \otimes_R H_\bullet(x_1, \dots, x_k)$, whence the result. □

Proposition 2.12. *Let $\pi\colon S \to R$ be an epimorphism of Noetherian local (graded) rings, where S is regular. Set $K := R/\mathfrak{m}$ with $\mathfrak{m} \subset R$ the maximal (homogeneous) ideal, and $m := \operatorname{edim}(S) - \operatorname{edim}(R)$. Then for $i \in \mathbb{Z}$ we have*

$$\sum_{j=0}^{m} \binom{m}{j} b_{i-j}(R) = \dim_K \left(\operatorname{Tor}_i^S(K,R)\right).$$

Proof. The maximal (homogeneous) ideal of S is $\mathfrak{n} := \pi^{-1}(\mathfrak{m})$. Choose minimal generators $y_1, \ldots, y_k$ of $\mathfrak{n}$, $k = \operatorname{edim}(S)$. By Bruns and Herzog [5, Corollary 2.2.6], $y_1, \ldots, y_k$ form an S-regular sequence. Therefore $\mathcal{K}_\bullet(y_1, \ldots, y_k)$ provides a free resolution of K as an S-module (see Bruns and Herzog [5, Corollary 1.6.14]). It follows that

$$\operatorname{Tor}_i^S(K,R) = H_i\left(\mathcal{K}_\bullet(y_1, \ldots, y_k) \otimes_S R\right).$$

Set $x_i := \pi(y_1) = R \otimes_S y_i$. Then by Bruns and Herzog [5, Proposition 1.6.7(a)] we obtain that $\mathcal{K}_\bullet(x_1, \ldots, x_k) \cong R \otimes_S \mathcal{K}_\bullet(y_1, \ldots, y_k)$, thus $\dim_K \left(\operatorname{Tor}_i^S(K,R)\right) = \dim_K \left(H_i(x_1, \ldots, x_k)\right)$. By renumbering the x_i we can assume that $x_1, \ldots, x_{k'}$ minimally generate $\mathfrak{m}$, so $k - k' = m$. By Bruns and Herzog [5, Proposition 1.6.21] we have

$$H_i(x_1, \ldots, x_k) = \bigoplus_{j=0}^{m} R^{\binom{m}{j}} \otimes_R H_{i-j}(x_1, \ldots, x_{k'}).$$

This yields

$$\dim_K \left(H_i(x_1, \ldots, x_k)\right) = \sum_{j=0}^{m} \binom{m}{j} \dim_K \left(H_{i-j}(x_1, \ldots, x_{k'})\right) = \sum_{j=0}^{m} \binom{m}{j} b_{i-j}(R).$$

The result now follows. $\square$

From Proposition 2.12 it follows that for a Noetherian local graded ring R the $b_i(R)$ may be defined as in (3) the Introduction.

Lemma 2.13. *Let R be a Noetherian local graded ring and $\mathfrak{p} \in \operatorname{Spec}(R)$. Set $m := \operatorname{edim}(R) - \operatorname{edim}(R_\mathfrak{p}) - \dim(R/\mathfrak{p})$. Then $m \geq 0$ and for $i \in \mathbb{Z}$ we have*

$$\sum_{j=0}^{m} \binom{m}{j} b_{i-j}(R_\mathfrak{p}) \leq b_i(R).$$

Proof. By Proposition 2.1 we have an epimorphism $\pi\colon S := K[x_1, \ldots, x_k] \to R$ with $K := R_0$ and S a polynomial ring in $k := \operatorname{edim}(R)$ indeterminates. Let

$$\cdots \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow R \longrightarrow 0$$

be a minimal graded free resolution of R as an S-module. Then $\operatorname{rank}(F_i) = \dim_K \left(\operatorname{Tor}_i^S(K,R)\right)$ (see Eisenbud [7, Exercise A3.18]), hence by Proposition 2.12 we have $b_i(R) = \operatorname{rank}(F_i)$. With $\mathfrak{q} := \pi^{-1}(\mathfrak{p})$ we have $R_\mathfrak{p} \cong S_\mathfrak{q} \otimes_S R$ (see Eisenbud [7, Lemma 2.4]). By the flatness of localization the sequence

$$\cdots \longrightarrow S_\mathfrak{q} \otimes_S F_2 \longrightarrow S_\mathfrak{q} \otimes_S F_1 \longrightarrow S_\mathfrak{q} \otimes_S F_0 \longrightarrow R_\mathfrak{p} \longrightarrow 0$$

provides a free resolution of $R_\mathfrak{p}$ over $S_\mathfrak{q}$. Since $S_\mathfrak{q} \otimes_S F_i$ is free of rank $b_i(R)$, we obtain

$$\dim_L \left(\operatorname{Tor}_i^{S_\mathfrak{q}}(L, R_\mathfrak{p})\right) \leq b_i(R),$$

where $L := S_\mathfrak{q}/\mathfrak{q}_\mathfrak{q} = R_\mathfrak{p}/\mathfrak{p}_\mathfrak{p}$. The lemma now follows from Proposition 2.12 with m replaced by $\operatorname{edim}(S_\mathfrak{q}) - \operatorname{edim}(R_\mathfrak{p})$. But $S_\mathfrak{q}$ is regular, so

$$\operatorname{edim}(S_\mathfrak{q}) = \dim(S_\mathfrak{q}) = \operatorname{ht}(\mathfrak{q}) = k - \dim(S/\mathfrak{q}) = \operatorname{edim}(R) - \dim(R/\mathfrak{p}).$$

This completes the proof. $\square$

Lemma 2.14. *Let R be a Noetherian local graded ring with $K = R_0$ and $S = K[x_1, \dots, x_n]$ a polynomial ring. Then for all $i \in \mathbb{Z}$ we have*

$$b_i(S \otimes_K R) = b_i(R).$$

Proof. As in the proof of Lemma 2.13 we have a minimal graded free resolution

$$\cdots \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow R \longrightarrow 0$$

of R as a module over a polynomial ring $S' := K[y_1, \dots, y_k]$ with $k = \operatorname{edim}(R)$, and then $\operatorname{rank}(F_i) = b_i(R)$ holds. Tensoring this resolution over S' by $S \otimes_K S'$ (which is free as an S'-module), we obtain the minimal graded free resolution

$$\cdots \longrightarrow (S \otimes_K S')^{b_2(R)} \longrightarrow (S \otimes_K S')^{b_1(R)} \longrightarrow (S \otimes_K S')^{b_0(R)} \longrightarrow S \otimes_K R \longrightarrow 0.$$

Since $\operatorname{edim}(S \otimes_K R) = n + k$ by Lemma 2.5, this yields $b_i(S \otimes_K R) = b_i(R)$. □

Theorem 2.15. *For $m, r \geq 0$ and $i \in \mathbb{Z}$ define the property $\mathcal{P}_{m,i,r}$ by saying that $\mathcal{P}_{m,i,r}$ holds for a Noetherian (graded) local ring R if $l := m - \operatorname{edim}(R) + \dim(R) \geq 0$ and*

$$\sum_{j=0}^{l} \binom{l}{j} b_{i-j}(R) \leq r. \tag{5}$$

Then $\mathcal{P}_{m,i,r}$ is local and geometric over every perfect field.

Moreover, if $\operatorname{edim}(R) - \dim(R) \leq m$ for a Noetherian local graded ring R, then $\operatorname{Loc}_R(\mathcal{P}_{m,i,r})$ satisfies the conditions (G1) and (G2) from Definition 1.4 even if R_0 is not perfect.

Proof. First observe that $l \geq 0$ is equivalent to the property $\mathcal{P}_m$ defined in Proposition 2.6. Now the conditions (L1), (L2), and (L4) follow directly from Lemmas 2.10, 2.11, and 2.14. To prove (L3), let R be a Noetherian local graded domain such that $\mathcal{P}_{m,i,r}(R)$ holds, and let $\mathfrak{p} \in \operatorname{Spec}(R)$ be a maximal ideal. Thus the inequality (5) holds and we have to show that

$$\sum_{j'=0}^{l'} \binom{l'}{j'} b_{i-j'}(R_{\mathfrak{p}}) \leq r \tag{6}$$

with $l' := m - \operatorname{edim}(R_{\mathfrak{p}}) + \dim(R_{\mathfrak{p}})$. From Lemma 2.13 we know that

$$\sum_{j''=0}^{l''} \binom{l''}{j''} b_{i-j''}(R_{\mathfrak{p}}) \leq b_i(R)$$

with $l'' := \operatorname{edim}(R) - \operatorname{edim}(R_{\mathfrak{p}}) - \dim(R/\mathfrak{p})$. Therefore

$$\sum_{j=0}^{l} \binom{l}{j} \left(\sum_{j''=0}^{l''} \binom{l''}{j''} b_{i-j-j''}(R_{\mathfrak{p}}) \right) \leq \sum_{j=0}^{l} \binom{l}{j} b_{i-j}(R) \leq r.$$

We have $l + l'' = m - \operatorname{edim}(R_{\mathfrak{p}}) - \dim(R/\mathfrak{p}) + \dim(R) = l'$ since R is a domain. Hence $\sum_{j+j''=j'} \binom{l}{j}\binom{l''}{j''} = \binom{l'}{j'}$, and (6) follows. We have proved that $\mathcal{P}_{m,i,r}$ is local.

To prove that $\mathcal{P}_{m,i,r}$ is geometric, let R be a Noetherian local graded domain with $K := R_0$ and choose a polynomial ring S with $\dim(S) = \operatorname{edim}(R)$ such that R is an epimorphic image of S. Let

$$\cdots \xrightarrow{\varphi_2} F_2 \xrightarrow{\varphi_1} F_1 \xrightarrow{\varphi_0} F_0 \longrightarrow R \longrightarrow 0 \tag{7}$$

be a minimal graded free resolution of R as an S-module, so $b_i(R) = \operatorname{rank}(F_i)$ by Proposition 2.12. Take $\mathfrak{p} \in \operatorname{Spec}(R)$ and let $\mathfrak{q}$ be the preimage of $\mathfrak{p}$ in S. Tensoring (7) by $S_\mathfrak{q}$ yields a free resolution of $R_\mathfrak{p}$ as an $S_\mathfrak{q}$-module. On the other hand, let

$$\cdots \longrightarrow F_2' \longrightarrow F_1' \longrightarrow F_0' \longrightarrow R_\mathfrak{p} \longrightarrow 0 \tag{8}$$

be a *minimal* free resolution of $R_\mathfrak{p}$ over $S_\mathfrak{q}$. By Proposition 2.12 we have

$$\operatorname{rank}(F_i') = \sum_{j=0}^{l_\mathfrak{p}} \binom{l_\mathfrak{p}}{j} b_{i-j}(R_\mathfrak{p})$$

with $l_\mathfrak{p} := \operatorname{edim}(S_\mathfrak{q}) - \operatorname{edim}(R_\mathfrak{p})$. By Eisenbud [7, Theorem 20.2], the resolution (7) tensored with $S_\mathfrak{q}$ is isomorphic to the direct sum of (8) and a trivial complex. We write this direct sum as

$$\cdots \longrightarrow F_2' \oplus S_\mathfrak{q}^{m_1} \oplus S_\mathfrak{q}^{m_2} \longrightarrow F_1' \oplus S_\mathfrak{q}^{m_0} \oplus S_\mathfrak{q}^{m_1} \longrightarrow F_0' \oplus S_\mathfrak{q}^{m_0} \longrightarrow R_\mathfrak{p} \longrightarrow 0$$

with m_i non-negative integers. For a finitely generated $S_\mathfrak{q}$-module M we write $\mu(M) := \dim_{S_\mathfrak{q}/\mathfrak{q}_\mathfrak{q}}(M/\mathfrak{q}_\mathfrak{q} M)$ for the size of a minimal generating system. Since the above complex is isomorphic to (7) tensored with $S_\mathfrak{q}$, we obtain $\mu\left(\operatorname{coker}(S_\mathfrak{q} \otimes \varphi_i)\right) = \operatorname{rank}(F_i') + m_{i-1}$ and $\operatorname{rank}(F_i') + m_i + m_{i-1} = b_i(R)$. Thus we have

$$\sum_{j=0}^{l_\mathfrak{p}} \binom{l_\mathfrak{p}}{j} b_{i-j}(R_\mathfrak{p}) = \mu\left(\operatorname{coker}(S_\mathfrak{q} \otimes \varphi_i)\right) + \mu\left(\operatorname{coker}(S_\mathfrak{q} \otimes \varphi_{i-1})\right) - b_{i-1}(R). \tag{9}$$

But for any value of j the locus of all $\mathfrak{q} \in \operatorname{Spec}(S)$ such that $\mu\left(\operatorname{coker}(S_\mathfrak{q} \otimes \varphi_i)\right) \le j$ is open, given by some Fitting ideal of φ_i (see Bruns and Herzog [5, Lemma 1.4.7]). Thus from (9) we see that for any non-negative c the set of all $\mathfrak{p} \in \operatorname{Spec}(R)$ such that $\sum_{j=0}^{l_\mathfrak{p}} \binom{l_\mathfrak{p}}{j} b_{i-j}(R_\mathfrak{p})$ equals c is constructible. Since the Fitting ideals have the same generators when we replace R by $\bar{K} \otimes_K R$ (see Eisenbud [7, Corollary 20.5]), this set is tensored by $\bar{K}$ if we pass from R to $\bar{K} \otimes_K R$. Now $\mathcal{P}_{m,i,r}$ holds for $R_\mathfrak{p}$ if and only if $l := m - \operatorname{edim}(R_\mathfrak{p}) + \dim(R_\mathfrak{p}) \ge 0$ and

$$\sum_{j=0}^{l} \binom{l}{j} b_{i-j}(R_\mathfrak{p}) \le r.$$

Since R and S are domains, we have

$$l - l_\mathfrak{p} = m + \dim(R_\mathfrak{p}) - \operatorname{edim}(S_\mathfrak{q}) = m + \dim(R) - \operatorname{edim}(R),$$

which does not depend on $\mathfrak{p}$. The identity $\sum_{j+j'=k} \binom{l-l_\mathfrak{p}}{j'}\binom{l_\mathfrak{p}}{j} = \binom{l}{k}$ (which also holds if $l - l_\mathfrak{p}$ is negative) leads to

$$\sum_{j=0}^{l} \binom{l}{j} b_{i-j}(R_\mathfrak{p}) = \sum_{j'=0}^{i} \binom{l - l_\mathfrak{p}}{j'} \left(\sum_{j=0}^{l_\mathfrak{p}} \binom{l_\mathfrak{p}}{j} b_{i-j-j'}(R_\mathfrak{p}) \right).$$

Since the set of $\mathfrak{p} \in \operatorname{Spec}(R)$ where the inner sum of the right hand side takes any specific value is constructible, the same is true for the set of $\mathfrak{p}$ where the whole sum is at most r. But the intersection of this set with $\operatorname{Loc}_R(\mathcal{P}_m)$ is $\operatorname{Loc}_R(\mathcal{P}_{m,i,r})$. We have also seen by that $\operatorname{Loc}_{R \otimes_K \bar{K}}(\mathcal{P}_{m,i,r}) = \operatorname{Loc}_R(\mathcal{P}_{m,i,r})_{\bar{K}}$. Thus (G1) and (G2) from Definition 1.4 are satisfied.

As for the last statement, we have seen that the perfectness of R_0 is only needed to ensure that the conditions (G1) and (G2) hold for $\operatorname{Loc}_R(\mathcal{P}_m)$. But if $\operatorname{edim}(R) - \dim(R) \ge m$, then $\mathfrak{p} \in \operatorname{Loc}_R(\mathcal{P}_m)$ holds for all $\mathfrak{p} \in \operatorname{Spec}(R)$ by Lemma 2.4. □

From Theorem 2.15, Theorem A(c) follows with Theorem 1.6. Since $b_i(R) \le \sum_{j=0}^{l} \binom{l}{j} b_{i-j}(R)$ holds in the situation of Theorem 2.15, it also follows that $b_i\left(K[V]^{G_U}\right) \le b_i(K[V]^G)$ for all subspaces $U \le V$.

2.3 The complete intersection, Cohen-Macaulay, and Gorenstein properties

The properties of being a polynomial ring or a hypersurface were already shown to be local and geometric in Section 2.1. Now we consider the properties of being a complete intersection, Gorenstein, or Cohen-Macaulay. We will show how these properties and the numbers measuring the deviation from them can be expressed in terms of Betti numbers and thus of the $\mathcal{P}_{m,i,r}$ from Theorem 2.15. As a corollary, it follows that the properties are local and geometric. The openness of the Gorenstein and complete intersection loci was already proved by Greco and Marinari [9]. For the Cohen-Macaulay locus, see also Matsumura [24, Exercise 24.2].

Recall that a Noetherian local (graded) ring R is called a complete intersection if it is the quotient of a regular (graded) ring modulo an ideal generated by a (homogeneous) regular sequence. This is equivalent to $b_1(R) = \operatorname{edim}(R) - \dim(R)$ (see Bruns and Herzog [5, Theorem 2.3.3] for the local case; in the global case it is easy to see that $R = K[x_1, \ldots, x_k]/I$ is a complete intersection if and only if I is generated by $k - \dim(R)$ homogeneous relations.) Let us call

$$\operatorname{cidef}(R) := b_1(R) - \operatorname{edim}(R) + \dim(R)$$

the **complete intersection defect** of R. By Krull's principal ideal theorem, this number is nonnegative.

Proposition 2.16. *For each $r \geq 0$, the property "$\operatorname{cidef}(R) \leq r$" is local and geometric.*

Proof. It follows from the definition of the $b_i(R)$ that $b_0(R) = 1$ for all R. Thus if $\operatorname{edim}(R) - \dim(R) \leq m$, then $\operatorname{cidef}(R) \leq r$ is equivalent with the property $\mathcal{P}_{m,1,r+m}(R)$ from Theorem 2.15. Thus $\operatorname{cidef}(R) \leq r$ holds if and only if $\mathcal{P}_{m,1,r+m}(R)$ holds for some m. This shows that "$\operatorname{cidef}(R) \leq r$" is local. For a Noetherian local graded domain R with $m := \operatorname{edim}(R) - \dim(R)$, we have $\operatorname{edim}(R_x) - \dim(R_x) \leq m$ for $x \in \operatorname{Spec}(R)$ by Lemma 2.4, and hence $\operatorname{Loc}_R(\operatorname{cidef} \leq r) = \operatorname{Loc}_R(\mathcal{P}_{m,1,r+m})$. By Theorem 2.15, this locus satisfies the conditions (G1) and (G2) from Definition 1.4 even if R_0 is not perfect. This completes the proof. □

For a Noetherian local (graded) ring R with maximal (homogeneous) ideal $\mathfrak{m}$ we write $\operatorname{depth}(R)$ for the maximal length of an R-regular sequence whose elements lie in $\mathfrak{m}$. R is Cohen-Macaulay if and only if $\operatorname{depth}(R) = \dim(R)$, so we define

$$\operatorname{cmdef}(R) := \dim(R) - \operatorname{depth}(R)$$

as the **Cohen-Macaulay defect** of R. The Cohen-Macaulay defect can be expressed in terms of the Betti numbers $b_i(R)$ as follows: if $\operatorname{hdim}(R) := \max\{i \in \mathbb{Z} \mid b_i(R) \neq 0\}$, then

$$\operatorname{cmdef}(R) = \operatorname{hdim}(R) - \operatorname{edim}(R) + \dim(R)$$

(see Bruns and Herzog [5, Theorem 1.6.17]).

Proposition 2.17. *For each $r \geq 0$, the property "$\operatorname{cmdef}(R) \leq r$" is local and geometric.*

Proof. If $\operatorname{edim}(R) - \dim(R) \leq m$, then $\operatorname{cmdef}(R) \leq r$ holds if and only if $\mathcal{P}_{m,i,0}(R)$ holds for all $i > r+m$. If R is an epimorphic image of a regular ring, then by Proposition 2.12 this is equivalent to $\mathcal{P}_{m,r+m+1,0}(R)$. But we can always assume that R is an epimorphic image of a regular ring, since in the local case $\mathcal{P}_{m,i,0}(R)$ is equivalent to the same condition on $\widehat{R}$. But by Cohen's structure theorem (see Bruns and Herzog [5, Theorem A.21]), $\widehat{R}$ is an epimorphic image of a regular ring. Now the result follows as in the proof of Proposition 2.16. □

Example 2.18. Consider the following linear groups given by Nakajima [26] as examples of groups generated by transvections whose invariant rings are not Cohen-Macaulay. Let $m \geq 3$ be an integer, set $n := 2m + 1$ and take G to be the group of all $n \times n$ matrices over a finite field $K = \mathbb{F}_q$ of the form

$$\sigma_{\alpha_0,\ldots,\alpha_m} := \left(\begin{array}{cccc|ccc} 1 & & & & & & \\ & 1 & & & & 0 & \\ & & \ddots & & & & \\ & & & 1 & & & \\ \hline \alpha_0 & \alpha_1 & & & 1 & & \\ \vdots & & \ddots & & & \ddots & \\ \alpha_0 & & & \alpha_m & & & 1 \end{array}\right) \tag{10}$$

with $\alpha_0, \ldots, \alpha_m \in K$. For the point $x = (-1, 1, \ldots, 1) \in V := K^n$, the point stabilizer is $G_x = \{\sigma_{\alpha,\ldots,\alpha} \mid \alpha \in K\}$. Since $\operatorname{rank}(\sigma_{\alpha,\ldots,\alpha} - \mathrm{id}) = m \geq 3$ for $\alpha \neq 3$, G_x is not generated by bireflections. Since G_x is a p-group, it follows by Kemper [19, Corollary 3.7] that $K[V]^{G_x}$ is not Cohen-Macaulay. Therefore by Theorem B, $K[V]^G$ is not Cohen-Macaulay. See Kemper [18, Beispiel 5.17] for a detailed computation of the non-Cohen-Macaulay locus. $\triangleleft$

Now we turn to the Gorenstein property. The **type** of a Cohen-Macaulay local (graded) ring R with maximal (homogeneous) ideal $\mathfrak{m}$ and $n := \dim(R)$ is defined as

$$r(R) := \dim_{R/\mathfrak{m}} \left(\operatorname{Ext}_R^n(R/\mathfrak{m}, R)\right).$$

If R is not Cohen-Macaulay, we formally set $r(R) := \infty$. A Noetherian local ring R is Gorenstein if it is of type 1 (see Bruns and Herzog [5, Theorem 3.20.10]). A Noetherian ring R is Gorenstein if all localizations $R_\mathfrak{p}$ at maximal ideals are Gorenstein (see Bruns and Herzog [5, Definition 3.1.18]). The following proposition tells us how the type and thus the Gorenstein property can be expressed in terms of the Betti numbers.

Lemma 2.19. *Let R be a Noetherian local (graded) ring which is Cohen-Macaulay. Then with $m := \operatorname{edim}(R) - \dim(R)$ we have*

$$r(R) = b_m(R).$$

Proof. Let $\mathfrak{m}$ be the maximal (homogeneous) ideal of R and write $K := R/\mathfrak{m}$. If R is local, then by Lemma 2.10 and Herzog and Kunz [12, Lemma 1.22] we have $b_m(\widehat{R}) = b_m(R)$ and $r(\widehat{R}) = r(R)$, so we may assume that R is complete. By Cohen's structure theorem, R is an epimorphic image of a regular local ring S. On the other hand, if R is a local graded ring, it is the epimorphic image of a polynomial ring S. We may assume that $\dim(S) = \operatorname{edim}(R)$ (see Proposition 2.1 and Bruns and Herzog [5, p. 72]). By Herzog and Kunz [12, Lemma 1.22(b)] (which holds for local graded Cohen-Macaulay rings as well as local Cohen-Macaulay rings), we have $r(R) = \dim_K \left(\operatorname{Ext}_S^n(K, R)\right)$, where $n := \dim(R)$. Choose minimal generators $x_1, \ldots, x_k$ of the maximal (homogeneous) ideal of S, so $k = \operatorname{edim}(R)$. Then the Koszul complex $\mathcal{K}_\bullet(x_1, \ldots, x_k)$ provides a free resolution of K over S, so

$$\operatorname{Ext}_S^i(K, R) = H_i\left(\operatorname{Hom}_S\left(\mathcal{K}_\bullet(x_1, \ldots, x_k), R\right)\right).$$

From the self-duality of the Koszul complex we obtain $H_i\left(\operatorname{Hom}_S\left(\mathcal{K}_\bullet(x_1, \ldots, x_k), R\right)\right) \cong H_{k-i}\left(\mathcal{K}_\bullet(x_1, \ldots, x_k) \otimes_S R\right)$ (see Bruns and Herzog [5, Proposition 1.6.10(d)]). In particular, $r(R) = b_{k-n}(R)$. But $k - n = m$. $\square$

Proposition 2.20. *For all positive integers k, the property of having type at most k is local and geometric. In particular, the Gorenstein property is local and geometric.*

Proof. By Lemma 2.19 and the proof of Proposition 2.17, a Noetherian local (graded) ring R with $m := \operatorname{edim}(R) - \dim(R)$ is of type $r(R) \leq k$ if and only if $b_m(R) \leq k$ and $b_{m+1}(R) = 0$. It is easy to check that this is equivalent to the condition that for some (and then for all) $m \geq \operatorname{edim}(R) - \dim(R)$ the properties $\mathcal{P}_{m,m,k}(R)$ and $\mathcal{P}_{m,m+1,0}(R)$ hold. This yields the first statement.

For the second statement, we only need to show that a Noetherian local graded ring R with maximal homogeneous ideal $\mathfrak{m}$ is Gorenstein if and only if R is of type 1. But by the locality, the latter condition is equivalent to saying that $R_\mathfrak{m}$ is of type 1, i.e., $R_\mathfrak{m}$ is Gorenstein. Moreover, by Bruns and Herzog [5, Exercise 3.6.20(c)] $R_\mathfrak{m}$ is Gorenstein if and only if R is Gorenstein. This completes the proof. □

We have now proved all assertions in Theorems A–C.

3 The Cohen-Macaulay and Buchsbaum properties

In this section we use Theorem C to obtain bounds on the dimension of the non-Cohen-Macaulay locus in $K[V]^G$. From these, we deduce that $K[V]^G$ is Buchsbaum if and only if it is Cohen-Macaulay.

3.1 The non-Cohen-Macaulay locus

By Proposition 2.17, the property "$\operatorname{cmdef}(R) \le m$" is local and geometric. It follows by Corollary 1.11 that the locus $\operatorname{Loc}_{K[V]^G}(\operatorname{cmdef} > m)$ is closed. The next theorem gives an upper and a lower bound for the dimension of this locus, which for $m = 0$ is the non-Cohen-Macaulay locus.

Theorem 3.1. *Let $G \le \mathrm{GL}(V)$ be a finite linear group acting on a finite dimensional vector space V over a field K. Let m be a non-negative integer such that $\operatorname{cmdef}\left(K[V]^G\right) > m$. Then*

$$0 < \dim\Big(\operatorname{Loc}_{K[V]^G}(\operatorname{cmdef} > m)\Big) < \dim_K(V) - m - 2.$$

In particular, the non-Cohen-Macaulay locus of $K[V]^G$ is either empty, or it has dimension at least one and codimension at least 3.

Proof. Write $X := \operatorname{Spec}(K[V])$ and $Y := \left\{x \in X \mid \operatorname{cmdef}\left(K[V]^{G_x}\right) > m\right\}$. Theorem 1.9 and Proposition 2.17 yield

$$\pi_G^{-1}\left(\operatorname{Loc}_{K[V]^G}(\operatorname{cmdef} > m)\right) = Y,$$

and the finiteness of π_G yields that $\dim\left(\operatorname{Loc}_{K[V]^G}(\operatorname{cmdef} > m)\right) = \dim(Y)$ (see Eisenbud [7, Corollary 9.3]). Let $P \le G$ be a Sylow p-subgroup of G with $p := \operatorname{char}(K)$. The P-orbit of a vector $0 \ne v \in V$ spans a finite-dimensional vector space over $\mathbb{F}_p$. Since P is a p-group, there exists a vector $0 \ne x$ from this space which is fixed by P. Hence $P \subseteq G_x$. Since the index of G_x is not divisible by p, it follows that $\operatorname{depth}\left(K[V]^{G_x}\right) \le \operatorname{depth}\left(K[V]\right)$ (see Kemper [18, Proposition 1.21]), so $\operatorname{cmdef}\left(K[V]^{G_x}\right) > m$. It follows that x lies in Y and therefore also $K \cdot x \subseteq Y$. This yields the lower bound for the dimension.

Y is a union of subspaces X^H with $H \le G$ subgroups such that $\operatorname{cmdef}\left(K[V]^{G_x}\right) > m$. Let $P' \le H$ be a Sylow p-subgroup of such an H. Then again $\operatorname{cmdef}\left(K[V]^{P'}\right) > m$. But by Ellingsrud and Skjelbred [8] we have

$$\operatorname{depth}\left(K[V]^{P'}\right) \ge \min\left\{\dim_K(V^{P'}) + 2, \dim_K(V)\right\},$$

so $\operatorname{cmdef}\left(K[V]^{P'}\right) \le \max\left\{\dim_K(V) - \dim_K(V^{P'}) - 2, 0\right\}$. Therefore $\operatorname{cmdef}\left(K[V]^{P'}\right) > m$ implies $m < \dim_K(V) - \dim_K(V^{P'}) - 2$ and hence

$$\dim_K(V^H) \le \dim_K(V^{P'}) < \dim_K(V) - m - 2.$$

This yields the upper bound. □

Remark. (a) Using the same arguments as in the proof of Theorem 3.1, with Theorem 5 from Landweber and Stong [22] instead of Ellingsrud and Skjelbred's result, one obtains that the singular locus in $K[V]^G$ has codimension at least 2. This is well-known and also follows from the fact that $K[V]^G$ is a normal domain.

(b) The following example shows that it may happen that the non-Gorenstein locus of an invariant ring consists of only one point and has codimension 2. Consider the group generated by $\left(\begin{smallmatrix} \zeta_3 & 0 \\ 0 & \zeta_3 \end{smallmatrix}\right)$, where $\zeta_3 \in K$ is a primitive third root of unity. $K[V]^G$ is generated by the four monomials of degree 3 and has the Hilbert series

$$H\left(K[V]^G, t\right) = \frac{1 + 2t^3}{(1 - t^3)^2}.$$

Therefore $K[V]^G$ is not Gorenstein by Stanley [29, p. 503]. Since all non-zero vectors have the trivial group as point stabilizer, we obtain $\pi_G(0)$ as the non-Gorenstein locus. Thus none of the bounds from Theorem 3.1 can be extended to the locus where a stronger property occurring in (1) in the Introduction is violated.

(c) The following example shows that the bounds from Theorem 3.1 cannot be sharpened. Let $G = Z_p$ be a cyclic group of order $p := \operatorname{char}(K)$ and let V_n be the n-dimensional indecomposable KG-module with $4 \le n \le p$. Then by Ellingsrud and Skjelbred [8] we have $\operatorname{depth}\left(K[V]^G\right) = 3$, so $\operatorname{cmdef}\left(K[V]^G\right) = n - 3$. Thus

$$\dim\left(\operatorname{Loc}_{K[V]^G}(\operatorname{cmdef} > n-4)\right) = \dim_K(V^G) = 1 = n - (n-4) - 3.$$

Thus both bounds from Theorem 3.1 are simultaneously reached. $\triangleleft$

3.2 Buchsbaum invariant rings are Cohen-Macaulay

Let R be a Noetherian local ring with maximal ideal $\mathfrak{m}$. A sequence $x_1, \ldots, x_r \in \mathfrak{m}$ is called **weakly regular** if for each $i = 1, \ldots, r$

$$(x_1, \ldots, x_{i-1}) : (x_i) = (x_1, \ldots, x_{i-1}) : \mathfrak{m},$$

i.e., if for every $y \in R$ such that yx_i lies in the ideal $(x_1, \ldots, x_{i-1})$ generated by $x_1, \ldots, x_{i-1}$ and for every $x \in \mathfrak{m}$, xy lies in $(x_1, \ldots, x_{i-1})$. Clearly $x_1, \ldots, x_r$ is weakly regular if it is regular. A sequence $x_1, \ldots, x_r \in \mathfrak{m}$ is called a **system of parameters** if $r = \dim(R)$ and $\dim\left(R/(x_1, \ldots, x_r)\right) = 0$. R is called **Buchsbaum** if every system of parameters is weakly regular (see Stückrad and Vogel [31, Definition 1.5]). It follows from the unmixedness theorem (see Bruns and Herzog [5, Theorem 2.1.6]) that in a Cohen-Macaulay ring every system of parameters is a regular sequence. Hence if R is Cohen-Macaulay, it is also Buchsbaum. A Noetherian local graded ring R with maximal homogeneous ideal $\mathfrak{m}$ is called Buchsbaum if $R_{\mathfrak{m}}$ is Buchsbaum (see Stückrad and Vogel [31, Definition 3.1]). In the graded situation it is also clear that Cohen-Macaulay implies Buchsbaum.

One reason why the Buchsbaum property is useful is that if R is a Noetherian local (graded) Buchsbaum ring with maximal (homogeneous) ideal $\mathfrak{m}$, then every system of parameters $x_1, \ldots, x_r \in \mathfrak{m}$ "measures the depth", i.e., $\operatorname{depth}(R)$ is the maximal integer k such that $x_1, \ldots, x_k$ is regular (see Campbell et al. [6, Proposition 25]). This follows easily from the definition and from the fact that all maximal regular sequences have equal length. This property of Buchsbaum rings was used by Campbell et al. [6] and Kemper [20] to prove that in many cases the Buchsbaum property and the Cohen-Macaulay property of $K[V]^G$ are equivalent. We will use the following lemma to prove that this is in fact always the case.

Lemma 3.2 (Stückrad and Vogel [31, Chapter I, Corollary 1.11])**.** *Let R be a Noetherian local ring with maximal ideal $\mathfrak{m}$. If R is Buchsbaum, then for all $\mathfrak{p} \in \operatorname{Spec}(R)$ with $\mathfrak{p} \neq \mathfrak{m}$, the localization $R_\mathfrak{p}$ is Cohen-Macaulay.*[1]

I thank Jürgen Herzog for bringing this lemma to my attention. We obtain the following consequence.

Corollary 3.3. *Let R be a Noetherian local graded domain with maximal homogeneous ideal $\mathfrak{m}$. If R is Buchsbaum, the $R_\mathfrak{p}$ is Cohen-Macaulay for all $\mathfrak{p} \in \operatorname{Spec}(R) \setminus \{\mathfrak{m}\}$.*

Proof. By way of contradiction, assume that $R_\mathfrak{p}$ is not Cohen-Macaulay for a $\mathfrak{p} \in \operatorname{Spec}(R) \setminus \{\mathfrak{m}\}$. Since the non-Cohen-Macaulay locus is a closed cone (see Matsumura [24, Exercise 24.2] or Kemper [18, Satz 5.4]), it is given by a homogeneous radical ideal $I \subseteq R$. Since $I \subseteq \mathfrak{p}$, I is a proper subideal of $\mathfrak{m}$. I is the intersection of the minimal primes lying above it and all minimal primes are homogeneous (see Eisenbud [7, Proposition 3.12]), hence there exists a homogeneous ideal $\mathfrak{q} \subset R$ such that $I \subseteq \mathfrak{q} \subsetneqq \mathfrak{m}$. Thus $R_\mathfrak{q}$ is not Cohen-Macaulay. But $R_\mathfrak{q} = (R_\mathfrak{m})_{\mathfrak{q}_\mathfrak{m}}$ and $\mathfrak{q}_\mathfrak{m} \neq \mathfrak{m}_\mathfrak{m}$. Thus Lemma 3.2 yields that $R_\mathfrak{m}$ is not Buchsbaum, a contradiction. □

The condition (L3) from Definition 1.4 follows for the Buchsbaum property from Corollary 3.3. Moreover, (L2) is satisfied by definition, and (L1) is Lemma 1.13 in Stückrad and Vogel [31]. It is probably true that the Buchsbaum property is local, but we do not need this result here.

Theorem 3.4. *Let $G \leq \operatorname{GL}(V)$ be a finite linear group acting on a finite dimensional vector space V over a field K. Then the invariant ring $K[V]^G$ is Buchsbaum if and only if it is Cohen-Macaulay.*

Proof. If $K[V]^G$ is Cohen-Macaulay, it is certainly Buchsbaum. Conversely, if $K[V]^G$ is not Cohen-Macaulay, then by Theorem 3.1 the non-Cohen-Macaulay locus of $K[V]^G$ has positive dimension. Therefore there exists $\mathfrak{p} \in \operatorname{Spec}\left(K[V]^G\right)$, $\mathfrak{p} \neq K[V]^G_+$, such that $K[V]^G_\mathfrak{p}$ is not Cohen-Macaulay. By Corollary 3.3, this implies that $K[V]^G$ is not Buchsbaum. □

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[1] In fact, the conclusion of the lemma holds under the weaker hypothesis the local cohomology modules $H^i_\mathfrak{m}(R)$ are of finite length for $i \neq \dim(R)$ (see Stückrad and Vogel [31, Appendix, Proposition 16]).

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