# The Cohen–Macaulay Property and Depth in Invariant Theory

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January 9, 2012

**Abstract**

This article gives a survey of results about the Cohen–Macaulay property and the depth of invariant rings.

## Introduction

The main object of study in invariant theory is the invariant ring of a given group action. Typical questions about such an invariant ring are: Can it be finitely generated? How can generators be obtained algorithmically? To what extent can group orbits be separated by invariants? What ring-theoretic properties does the invariant ring have, and how do they relate to properties of the group action? This paper deals with the last question and focuses on the Cohen–Macaulay property and the depth. We will usually (but not always) restrict to the case of a finite group acting linearly on a finite-dimensional vector space. This paper will present methods using group cohomology to prove results about the Cohen–Macaulay property and depth of invariant rings. These methods were for the most part developed around the last turn of the century.

It is well known that in the nonmodular case (i.e., when the group order is not divisible by the characteristic $p$ of the ground field), the invariant ring is always Cohen–Macaulay. So the interesting case is the modular case. Before the development of the above-mentioned cohomological methods, only sporadic results and examples in the modular case were known. The best of these is a result by Ellingsrud and Skjelbred [7], who considered the case of cyclic groups and obtained an explicit formula for the depth of the invariant ring. Another result by Campbell et al. [4] tells us that if $G$ is a $p$-group, then vector invariants of three copies are never Cohen–Macaulay.

The first section of this paper is devoted to the nonmodular case. We generalize the above-mentioned result that invariant rings in this case are always

Cohen–Macaulay. The second section introduces the cohomological methods for the study of the Cohen–Macaulay property and depth. The main results in the case of finite groups are presented in Section 3. These include a result on vector invariants and a result saying that the group is generated by certain types of elements, which both go back to joint work with Nikolai Gordeev. Finally in the last section some results where the depth is determined precisely are discussed.

We will use the following notation. For a Noetherian ring $R$ we consider the **Cohen–Macaulay defect**

$$\operatorname{def}(R) := \sup\{\dim(R_P) - \operatorname{depth}(R_P) \mid P \in \operatorname{Spec}(R)\} \in \mathbb{N}_0 \cup \{\infty\},$$

which measures the deviation of $R$ from being Cohen–Macaulay. (In fact, for this definition it suffices to assume that $R_P$ is always finite-dimensional.) This number is particularly interesting in the case that $R = \bigoplus_{i \geq 0} R_i$ is a graded ring with $R_0$ a field, which occurs for example when $R$ is the invariant ring of a group acting linearly on a vector space. In this case we can use Noether normalization to obtain a graded subalgebra $A \subseteq R$, isomorphic to a polynomial ring, such that $R$ is finitely generated as an $A$-module. The Auslander–Buchsbaum formula then tells us that $\operatorname{def}(R)$ is equal to the length of a minimal free resolution of $R$ as an $A$-module. So $\operatorname{def}(R)$ measures the homological complexity of $R$. In particular, $R$ is free as an $A$-module if and only if it is Cohen–Macaulay.

We will consider the standard situation of invariant theory, so $V = K^n$ will be a finite-dimensional vector space over a field $K$, and $G \subseteq \operatorname{GL}(V)$ will be a subgroup of the general linear group. We will often make the restriction that $G$ is finite or, more generally, algebraic. If not stated otherwise,

$$R := K[V] = K[x_1, \ldots, x_n]$$

will denote the polynomial ring on $V$, on which $G$ acts by $\sigma(f) = f \circ \sigma^{-1}$. (If $K$ is finite, the action is first defined on the dual $V^*$ of $V$ as above and then on $K[V]$ by homomorphic extension.) Moreover,

$$R^G := \{f \in R \mid \sigma(f) = f \text{ for all } \sigma \in G\}$$

will denote the **invariant ring**. This is the main object of interest in invariant theory. In this article, our interest focuses on the Cohen–Macaulay defect $\operatorname{def}(R^G)$. With this notation, the formula by Ellingsrud and Skjelbred [7] mentioned above can be stated as follows. If $G$ is a cyclic group with Sylow $p$-subgroup $P$ (where $p = \operatorname{char}(K)$), then

$$\operatorname{def}(R^G) = \max\{\operatorname{codim}(V^P) - 2, 0\}. \tag{0.1}$$

# 1 The nonmodular case

The nonmodular case in invariant theory of finite groups is the case where the group order $|G|$ is finite and not divisible by the characteristic of $K$. This is the case where the results are nicest. For example, $R^G$ is always Cohen–Macaulay in this case. We will present a proof of this by giving a more general result which, to the best of the author's knowledge, has not yet appeared in the literature in this generality.

**Theorem 1.1.** *Let $R$ be a commutative ring with unity and let $G \subseteq \operatorname{Aut}(R)$ be a group of ring-automorphisms of $R$. Furthermore, let $H \subseteq G$ be a subgroup such that the index $(G : H)$ is finite and invertible in $R$, and assume that $R^H$ is Noetherian. Then*

$$\operatorname{def}(R^G) \leq \operatorname{def}(R^H).$$

*Proof.* We may assume $\operatorname{def}(R^H) < \infty$. Choose a system $\sigma_1, \ldots, \sigma_n$ of left coset representatives of $H$ in $G$. Since every $a \in R^H$ satisfies the equation $\prod_{i=1}^n (x - \sigma_i(a)) \in R^G[x]$, $R^H$ is an integral over $R^G$. Let $Q, Q' \in \operatorname{Spec}(R^H)$ such that

$$R^G \cap Q = R^G \cap Q'.$$

Then for $a \in Q$ we have $\prod_{i=1}^n \sigma_i(a) \in R^G \cap Q \subseteq Q'$, so there exists $i$ with $a \in \sigma_i^{-1}(Q')$. Using the prime avoidance lemma, we conclude that there exists $i$ such that $Q \subseteq \sigma_i^{-1}(Q')$. Since $Q$ and $Q'$ lie over the same prime ideal in $R^G$, this implies

$$Q' = \sigma_i(Q).$$

We claim that going down holds for the extension $R^G \subseteq R^H$. Let $P \in \operatorname{Spec}(R^G)$ and let $Q' \in \operatorname{Spec}(R^H)$ such that $P \subseteq Q'$. There exist $Q, \widetilde{Q} \in \operatorname{Spec}(R^H)$ such that

$$R^G \cap Q = P, \quad R^G \cap \widetilde{Q} = R^G \cap Q', \quad \text{and} \quad Q \subseteq \widetilde{Q}.$$

By the above, there exists $i$ with $Q' = \sigma_i(\widetilde{Q})$, so $\sigma_i(Q) \subseteq Q'$ and $R^G \cap \sigma_i(Q) = P$. So indeed going down holds.

Now let $P \in \operatorname{Spec}(R^G)$. We need to show that

$$\operatorname{depth}\left(R_P^G\right) \geq \dim(R_P^G) - \operatorname{def}(R^H).$$

Let $a_1, \ldots, a_m$ be a maximal $R^H$-regular sequence in $P$. Then every element of $P$ is contained in an associated prime ideal of $R^H/(a_1, \ldots, a_m)R^H$, so by prime avoidance, $P$ itself is contained in an associated prime ideal $Q$ of $R^H/(a_1, \ldots, a_m)R^H$. It is easy to see that as elements of $R_Q^H$, the $a_i$ form a maximal $R_Q^H$-regular sequence, so

$$m = \operatorname{depth}(R_Q^H) \geq \dim(R_Q^H) - \operatorname{def}(R^H) \geq \dim(R_P^G) - \operatorname{def}(R^H), \tag{1.1}$$

where the last inequality follows from going down (see Kemper [17, Corollary 8.14]). We claim that $a_1, \ldots, a_m$ is $R^G$-regular. So suppose that

$$b \cdot a_k = \sum_{j=1}^{k-1} b_j a_j$$

with $1 \leq k \leq m$ and $b, b_j \in R^G$. The $R^H$-regularity yields

$$b = \sum_{j=1}^{k-1} c_j a_j$$

with $c_i \in R^H$, so

$$b = \frac{1}{n}\sum_{i=1}^{n} \sigma_i(b) = \sum_{j=1}^{k-1} \left(\frac{1}{n}\sum_{i=1}^{n}\sigma_i(c_j)\right) a_j \in (a_1, \dots, a_m)R^G.$$

This proves the claim, so

$$m \leq \operatorname{grade}(P, R^G) \leq \operatorname{depth}(R^G_P),$$

where the second inequality follows from Bruns and Herzog [2, Proposition 1.2.10(a)]. Together with (1.1), this completes the proof. □

Returning to our standard situation where $R = K[V]$ and $G$ acts linearly on $V$, we state the following consequence of Theorem 1.1.

**Corollary 1.2** (Hochster and Eagon [12])**.** *Assume that $G$ is finite such that $|G|$ is not divisible by* $\operatorname{char}(K)$*. Then $R^G$ is Cohen–Macaulay.*

A further consequence of Theorem 1.1 is the result by Campbell et al. [3] that if $R^P$ is Cohen–Macaulay with $P \subseteq G$ a Sylow $p$-subgroup, $p = \operatorname{char}(K)$, then $R^G$ is also Cohen–Macaulay.

Recall that a linear algebraic group $G$ over an algebraically closed field $K$ is called **linearly reductive** if every $G$-**module** $V$ (i.e., every finite-dimensional $K$-vector space $V$ with a linear action given by a morphism $G \times V \to V$) is completely reducible. So a finite group $G$ is linearly reductive if and only if $|G|$ is not divisible by $\operatorname{char}(K)$. The following celebrated result is a generalization of Corollary 1.2.

**Theorem 1.3** (Hochster and Roberts [13])**.** *Let $G$ be a linearly reductive algebraic group over an algebraically closed field $K$ and let $V$ be a $G$-module. Then $K[V]^G$ is Cohen–Macaulay.*

## 2 A cohomological obstruction

We will now consider the more difficult nonmodular case of invariant theory and start by considering an example.

*Example 2.1.* Let $K$ be a field of positive characteristic $p$. The cyclic group $G = \langle\sigma\rangle \cong C_p$ of order $p$ acts on the polynomial ring $R := K[x_1, x_2, x_3, y_1, y_2, y_3]$ by

$$\sigma(x_i) = x_i \quad \text{and} \quad \sigma(y_i) = y_i + x_i.$$

We find invariants

$$x_i \quad (i = 1, 2, 3) \quad \text{and} \quad u_{i,j} := x_i y_j - x_j y_i \quad (1 \le i < j \le 3)$$

and the relation

$$x_1 u_{2,3} - x_2 u_{1,3} + x_3 u_{1,2} = \det \begin{pmatrix} x_1 & x_2 & x_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix} = 0.$$

Since $u_{1,2}$ does not lie in the ideal $(x_1, x_2)R^G$, this shows that $x_1, x_2, x_3$ do not form a $R^G$-regular sequence. On the other hand, the $x_i$ can be completed to a homogeneous system of parameters (by the invariants $y_i^p - x_i^{p-1} y_i$, for example), so it follows that $R^G$ is not Cohen–Macaulay. This is probably the most accessible example of a non-Cohen–Macaulay invariant ring. Of course we know from (0.1) that $\operatorname{def}(R^G) = 1$. ◁

In the above example, the invariants $x_i$ form an $R$-regular sequence, but fail to be $R^G$-regular. From a general investigation of this phenomenon, the cohomological criterion given in the following lemma emerged. We consider group cohomology $H^*(G, R)$ with values in $R$ and write

$$m := \inf \left\{ i > 0 \mid H^i(G, R) \neq 0 \right\} \in \mathbb{N} \cup \{\infty\}, \tag{2.1}$$

which we call the **cohomological connectivity** (see Fleischmann et al. [10]). This number is not always easily accessible, but in many cases it is.

**Lemma 2.2** (Kemper [14])**.** *Let $a_1, \ldots, a_r \in R^G$ be an $R$-regular sequence. If $r < m+2$ (with $m$ defined in (2.1)), then $a_1, \ldots, a_r$ is also $R^G$-regular. If $r = m+2$, then $a_1, \ldots, a_r$ is $R^G$-regular if and only if the map*

$$H^m(G, R) \to H^m(G, R^r)$$

*induced by the multiplication by $a_1, \ldots, a_r$ is injective.*

The last statement may be rephrased as follows: $a_1, \ldots, a_r$ fail to be $R^G$-regular if and only if there exists a nonzero $\alpha \in H^m(G, R)$ such that $a_i \alpha = 0$ for all $i$.

The case $m = 1$ of the lemma can be proved by elementary calculations. The general case can be proved by using the long exact sequence of cohomology and a Koszul complex.

*Example 2.3.* We reconsider Example 2.1 in the light of the above criterion. A nonzero cohomology class in $\alpha \in H^1(G, R)$ is given by the cocycle $G \to R,\ \sigma^j \mapsto j \in K$. So the cohomological connectivity is $m = 1$. For $i = 1, 2, 3$, the class $x_i \alpha \in H^1(G, R)$ is given by

$$G \to R,\ \sigma^j \mapsto j x_i = \sigma^j(y_i) - y_i,$$

which is a coboundary. So $x_i \alpha = 0$, and Lemma 2.2 tells us that $x_1, x_2, x_3$ are not $R^G$-regular. ◁

Lemma 2.2 is crucial for proving the general result presented in the following theorem. Before we state it, we recall that a linear algebraic group $G$ over an algebraically closed field $K$ is called **reductive** if it has no infinite normal, unipotent subgroup. Examples of reductive groups include all classical groups (such as $\mathrm{GL}_n(K)$, $\mathrm{SL}_n(K)$ and the symplectic and orthogonal groups) and all finite groups. Every linearly reductive group is reductive, and in characteristic 0 the converse holds. But in positive characteristic there is a wide gap between reductive groups and linearly reductive groups. The following result may be regarded as a converse to Theorem 1.3.

**Theorem 2.4** (Kemper [15])**.** *Let $G$ be a reductive algebraic group. If $G$ is not linearly reductive, there exists a $G$-module $V$ such that $K[V]^G$ is not Cohen-Macaulay.*

*Proof (sketch).* It is not hard to see that a linear algebraic group $G$ is linearly reductive if and only if $H^1(G, U) = 0$ for every $G$-module $U$. So under our hypotheses there exists a $G$-module $U$ with $H^1(G, U) \neq 0$. Choose a nonzero $\alpha \in H^1(G, U)$. Such a class $\alpha$ defines an exact sequence

$$0 \to U \to W \to K \to 0$$

of $G$-modules. (This can be seen by elementary considerations or, more abstractly, by interpreting $H^1(G, U) = \mathrm{Ext}^1_{KG}(K, U)$ as Yoneda Ext.) Dualizing the above sequence yields

$$0 \to K \to W^* \to U^* \to 0.$$

If $w \in (W^*)^G$ is the image of $1 \in K$, it turns out that $w \otimes \alpha$ is 0 as an element of $H^1(G, W^* \otimes U)$. Forming $V := W \oplus W \oplus W \oplus U^*$, we find three copies $a_1, a_2, a_3$ of $w$ in $R := K[V]$. As an element of $H^1(G, R)$, $\alpha$ remains nonzero, but $a_i\alpha = 0$. So by Lemma 2.2, the $a_i$ do not form an $R^G$-regular sequence. On the other hand, the $a_i$ can clearly be extended to a homogeneous parameter system of $R$. Since $G$ is reductive, it can be shown that they can also be extended to a homogeneous parameter system of $R^G$. Therefore $R^G$ is not Cohen–Macaulay. □

The reductivity hypothesis in Theorem 2.4 cannot be omitted. For example, every invariant ring of the additive group over $K = \mathbb{C}$ is Cohen–Macaulay. It may be worthwhile to mention in this context that, to the best of the author's knowledge, no example of a non-Cohen–Macaulay invariant ring $K[V]^G$ with $\mathrm{char}(K) = 0$ is known to date. An explicit version of Theorem 2.4 can be found in Kohls [18] (see [19] for results on the Cohen-Macaulay defect).

## 3 Traces and wild ramification

In Theorem 2.4 (and its proof) we have produced a tailor-made representation $V$ for a given group $G$ such that Lemma 2.2 could be used to prove that $K[V]^G$ is not Cohen–Macaulay. The goal of this section is to use Lemma 2.2 to deduce results on a given representation $V$ of a group $G$.

We will restrict to the case that $G$ is finite and its order is divisible by $p := \mathrm{char}(K)$. The question is which elements of $R^G$ annihilate cohomology classes from $H^i(G, R)$ with $i > 0$. One answer is the following: For a polynomial $f \in R$, define the **trace** as

$$\mathrm{Tr}(f) := \sum_{\sigma \in G} \sigma(f) \in R^G.$$

Then for $\alpha \in H^i(G, R)$ with $i > 0$ and $f \in R$ we have

$$\mathrm{Tr}(f) \cdot \alpha = \mathrm{cores}\left(f \cdot \mathrm{res}_{G,\{\mathrm{id}\}}(\alpha)\right) = 0, \tag{3.1}$$

where cores denotes the corestriction (see Evens [8, Proposition 4.2.4]).

Clearly the image $I := \mathrm{Tr}(R) \subseteq R^G$ of the trace map is an ideal. It is quite easy to determine its height. Since going down holds for $R^G \subseteq R$, the height of $I$ equals the height of the ideal $IR$ in $R$. So we consider the variety in $V$ determined by $I$. It is easy to see that for $x \in V$ the equivalence

$$\mathrm{Tr}(f)(x) = 0 \quad \text{for all } f \in R \quad \Longleftrightarrow \quad p \mid |G_x|$$

holds, where

$$G_x := \{\sigma \in G \mid \sigma(x) = x\}$$

denotes the point-stabilizer. So the variety determined by $I$ is the union of all $V^\sigma$ with $\sigma \in G$ of order $p$. We obtain

$$\mathrm{ht}\left(\mathrm{Tr}(R)\right) = \min\left\{\mathrm{codim}(V^\sigma) \mid \sigma \in G,\ \mathrm{ord}(\sigma) = p\right\} =: c. \tag{3.2}$$

So the height of the trace ideal is completely accessible. We can now prove the following result.

**Theorem 3.1.** *Let $G$ be finite. Then*

$$\mathrm{def}\left(R^G\right) \geq c - m - 1$$

*with $c$ and $m$ defined by (3.2) and (2.1).*

*Proof.* We may assume $m < \infty$. Let $P$ be an associated prime ideal of $H^m(G, R)$ as an $R^G$-module. We claim that

$$\mathrm{grade}\left(P, R^G\right) \leq m + 1. \tag{3.3}$$

Indeed, if there existed a regular sequence $a_1, \ldots, a_{m+2} \in P$, then the ideal in $R$ generated by the $a_i$ would have height $m+2$ (since going down holds for $R^G \subseteq R$), so the $a_i$ would form an $R$-regular sequence by the Cohen–Macaulay property of $R$. Applying Lemma 2.2 then shows that $a_1, \ldots, a_{m+2}$ is not $R^G$-regular after all. By Bruns and Herzog [2, Proposition 1.2.10(a)], there exists $Q \in \mathrm{Spec}(R^G)$ with $P \subseteq Q$ such that

$$\mathrm{depth}(R^G_Q) = \mathrm{grade}\left(P, R^G\right). \tag{3.4}$$

By (3.1), the trace ideal $\mathrm{Tr}(R)$ is contained in $P$ and therefore also in $Q$, so $\mathrm{ht}(Q) \geq c$ by (3.2). We obtain

$$\mathrm{def}(R^G) \geq \mathrm{ht}(Q) - \mathrm{depth}(R^G_Q) \geq c - m - 1,$$

where (3.3) and (3.4) were used for the last inequality. □

Now we can give a lower bound for the less accessible quantity $m$, the cohomological connectivity. In fact, if $|G| = p^a m$ with $(p, m) = 1$ and $a > 0$, then it can be shown that there exists a positive integer $r \le 2p^{a-1}(p-1)$ such that $H^r(G, \mathbb{F}_p)$ is nonzero. The argument uses the Evens norm in cohomology and can be found in the proof of Theorem 4.1.3 in Benson [1]. It follows that

$$m \le 2p^{a-1}(p-1) < 2|G|. \tag{3.5}$$

We obtain the following result on vector invariants, i.e., invariants of several copies of the same representation $V$.

**Corollary 3.2** (Gordeev and Kemper [11]). *Assume that $G$ is finite of order divisible by $p$. Then*

$$\lim_{k\to\infty} \operatorname{def}\left( K[\underbrace{V \oplus \cdots \oplus V}_{k \ \text{copies}}]^G \right) = \infty.$$

This follows from Theorem 3.1 and (3.5) since the number $c$ from (3.2) tends (linearly) to infinity when $V$ is replaced be the direct sum of $k$ copies of $V$. Corollary 3.2 tells us that the vector invariants in the modular case are getting worse and worse, in terms of homological complexity, as the number of copies increases.

We also obtain results that link the Cohen–Macaulay defect to the question by what type of elements $G$ can be generated. An element $\sigma \in \mathrm{GL}(V)$ is called a $k$-**reflection** if $\operatorname{codim}(V^\sigma) \le k$. So the 1-reflections are the identity and the pseudo reflections in the classical sense. In this context, a well-known result, attributed to Shephard, Todd, Chevalley, and Serre, says that if $R^G$ is isomorphic to a polynomial ring, then $G$ is generated by 1-reflections. (In the nonmodular case, the converse holds.) Concerning the Cohen–Macaulay defect, we can use our methods to deduce the following result.

**Theorem 3.3** (Gordeev and Kemper [11]). *Set $k := \operatorname{def}\left(R^G\right) + 2$. Then $G$ is generated by $k$-reflections and $p'$-elements (i.e., elements of order not divisible by $p$).*

*Proof (sketch).* Let $N \subseteq G$ be the (normal) subgroup generated by the $k$-reflections and $p'$-elements in $G$, and assume, by way of contradiction, that $N \subsetneq G$. Then $H^1(G/N, K) \ne 0$, so the image of the inflation map $H^1(G/N, R) \to H^1(G, R)$ is a nonzero submodule $M \subseteq H^1(G, R)$. As in the proof of Theorem 3.1, we choose an associated prime ideal $P$ of $M$ and find $Q \in \operatorname{Spec}(R^G)$ with $P \subseteq Q$ such that

$$\operatorname{depth}(R^G_Q) = \operatorname{grade}(P, R^G) \le 2.$$

On the other hand, it follows as (3.1) that every *relative trace*

$$\operatorname{Tr}_{N,G}(f) := \sum_{\sigma \in G/N} \sigma(f)$$

with $f \in R^N$ annihilates every element from $M$. So

$$\mathrm{Tr}_{N,G}(R^N) \subseteq P \subseteq Q.$$

As after (3.1), one can determine the variety in $V$ defined by $\mathrm{Tr}_{N,G}(R^N)$ and finds that this is the union of all $V^\sigma$ with $\sigma \in G$ such that $\sigma N \in G/N$ has order $p$ (see Fleischmann [9] or Campbell et al. [5, Theorem 7]). So

$$\mathrm{ht}(Q) \geq \mathrm{ht}(P) \geq \mathrm{ht}\left(\mathrm{Tr}_{N,G}(R^N)\right) = \min\left\{\mathrm{codim}(V^\sigma) \mid \sigma \in G,\ \mathrm{ord}(\sigma N) = p\right\} > k,$$

since $G \setminus N$ contains no $k$-reflections by the definition of $N$. Therefore

$$\mathrm{ht}(Q) - \mathrm{depth}(R_Q^G) > k - 2 = \mathrm{def}(R^G),$$

a contradiction. □

A special case of Theorem 3.3 says that if $G$ is a $p$-group and $R^G$ is Cohen–Macaulay, then $G$ is generated by 2-reflections (see Kemper [14]). This generalizes the result by Campbell et al. [4] mentioned in the Introduction. Unfortunately, the converse of Theorem 3.3 does not hold. In fact, there are examples of groups generated by 1-reflections such that the Cohen–Macaulay defect of $R^G$ is arbitrarily large.

# 4 Determining the Cohen–Macaulay defect

So far we have only achieved to establish lower bounds for the Cohen–Macaulay defect. But can anything be said about the exact value? For a given group $G \subseteq \mathrm{GL}(V)$, the invariant ring $R^G$ can be computed algorithmically (given enough time and memory space), and from this $\mathrm{def}(R^G)$ can be determined (see Derksen and Kemper [6, Chapter 3]. Apart from this, theoretical results on the precise value of the Cohen-Macaulay defect are rather sporadic. One is the formula (0.1) by Ellingsrud and Skjelbred [7]. Fleischmann et al. [10] proved the upper bound

$$\mathrm{def}\left(R^G\right) \leq \max\left\{\mathrm{codim}(V^P) - m - 1, 0\right\}, \tag{4.1}$$

where $P \subseteq G$ is a Sylow $p$-subgroup ($p = \mathrm{char}(K)$) and $m$ is the cohomological connectivity. In all instances of theoretical results where the Cohen–Macaulay defect of $R^G$ was determined, this bound turns out to be an equality. In fact,

$$\mathrm{def}\left(R^G\right) = \max\left\{\mathrm{codim}(V^P) - m - 1, 0\right\}$$

holds if

- $|G|$ is divisible by $p$ but not by $p^2$ (see Kemper [16, Theorem 3.1]; the determination of $m$ is hard in general),
- $|G|$ is divisible by $p$ but not by $p^2$ and acts by permutations of a basis of $V$ (see [16, Theorem 3.3], which gives a formula for $m$ is this case),

- $P$ is cyclic and $G$ is $p$-nilpotent, i.e., there exists a normal subgroup $N \subseteq G$ with $G/N \cong P$ (see Fleischmann et al. [10]; in this case $m = 1$),
- $G = \mathrm{SL}_2(\mathbb{F}_p)$ and $V$ is a symmetric power of the natural representation (see Shank and Wehlau [20]; here $m = 1$),
- more generally, $(G, V)$ is one of the cases dealt with in [16, Section 4] (then $m = 1$).

Notice that the result of Ellingsrud and Skjelbred is included in the one on $p$-nilpotent groups with a cyclic Sylow $p$-subgroup.

## References


[1] David J. Benson, *Representations and Cohomology II*, Cambridge Studies in Advanced Mathematics **31**, Cambridge Univ. Press, Cambridge 1991.

[2] Winfried Bruns, Jürgen Herzog, *Cohen-Macaulay Rings*, Cambridge University Press, Cambridge 1993.

[3] H. E. A. Campbell, I. P. Hughes, R. D. Pollack, *Rings of Invariants and p-Sylow Subgroups*, Canad. Math. Bull. **34(1)** (1991), 42–47.

[4] H. E. A. Campbell, A. V. Geramita, I. P. Hughes, R. J. Shank, D. L. Wehlau, *Non-Cohen-Macaulay Vector Invariants and a Noether Bound for a Gorenstein Ring of Invariants*, Canad. Math. Bull. **42** (1999), 155–161.

[5] H. E. A. Campbell, I. P. Hughes, G. Kemper, R. J. Shank, D. L. Wehlau, *Depth of Modular Invariant Rings*, Transformation Groups **5** (2000), 21–34.

[6] Harm Derksen, Gregor Kemper, *Computational Invariant Theory*, Encyclopaedia of Mathematical Sciences **130**, Springer-Verlag, Berlin, Heidelberg, New York 2002.

[7] Geir Ellingsrud, Tor Skjelbred, *Profondeur d'anneaux d'invariants en caractéristique p*, Compos. Math. **41** (1980), 233–244.

[8] Leonard Evens, *The Cohomology of Groups*, Oxford University Press, Oxford 1991.

[9] Peter Fleischmann, *Relative Trace Ideals and Cohen-Macaulay Quotients of Modular Invariant Rings*, in: P. Dräxler, G.O. Michler, C. M. Ringel, eds., *Computational Methods for Representations of Groups and Algebras, Euroconference in Essen, April 1-5 1997*, Progress in Mathematics **173**, Birkhäuser, Basel 1999.

[10] Peter Fleischmann, Gregor Kemper, R. James Shank, *Depth and Cohomological Connectivity in Modular Invariant Theory*, Trans. Amer. Math. Soc. **357** (2005), 3605–3621.

[11] Nikolai Gordeev, Gregor Kemper, *On the Branch Locus of Quotients by Finite Groups and the Depth of the Algebra of Invariants*, J. Algebra **268** (2003), 22–38.

[12] Melvin Hochster, John A. Eagon, *Cohen-Macaulay Rings, Invariant Theory, and the Generic Perfection of Determinantal Loci*, Amer. J. of Math. **93** (1971), 1020–1058.

[13] Melvin Hochster, Joel L. Roberts, *Rings of Invariants of Reductive Groups Acting on Regular Rings are Cohen-Macaulay*, Adv. in Math. **13** (1974), 115–175.

[14] Gregor Kemper, *On the Cohen-Macaulay Property of Modular Invariant Rings*, J. of Algebra **215** (1999), 330–351.

[15] Gregor Kemper, *A Characterization of Linearly Reductive Groups by their Invariants*, Transformation Groups **5** (2000), 85–92.

[16] Gregor Kemper, *The Depth of Invariant Rings and Cohomology,* with an appendix by Kay Magaard, J. of Algebra **245** (2001), 463–531.

[17] Gregor Kemper, *A Course in Commutative Algebra*, Graduate Texts in Mathematics **256**, Springer-Verlag, Berlin, Heidelberg 2011.

[18] Martin Kohls, *Non Cohen-Macaulay Invariant Rings of Infinite Groups*, J. Algebra **306** (2006), 591–609.

[19] Martin Kohls, *On the depth of invariant rings of infinite groups*, J. of Algebra **322** (2009), 210–218.

[20] R. James Shank, David L. Wehlau, *On the Depth of the Invariants of the Symmetric Power Representations of $SL_2(\mathbf{F}_p)$*, J. of Algebra **218** (1999), 642–653.