Morphisms and Constructible Sets: Making Two Theorems of Chevalley Constructive

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Abstract

Let $f: X \to Y$ be a morphism of affine varieties. We present an algorithm which computes the image $f(X)$ as a constructible set. The fibre dimensions $\dim\left(f^{-1}(y)\right)$ are also computed. More generally, images of constructible sets can be computed. Moreover, we present an algorithm which for any constructible subset $S \subseteq X$ computes a subset $U \subseteq S$ which is dense and open in the closure $\overline{S}$. The algorithms also apply to affine schemes of finite type over a field.

Introduction

The image of a morphism $f: X \to Y$ of varieties is, in general, neither closed nor open. By a theorem of Chevalley, it is, however, a constructible subset of Y (see Hartshorne [4, Exercise 3.18 and 3.19]). More generally, the image $f(S)$ of a constructible subset $S \subseteq X$ is a constructible subset of Y. Moreover, every constructible subset $S \subseteq X$ has a subset $U \subseteq S$ which is dense and open in the Zariski-closure $\overline{S}$. In particular, the image of a morphism has a subset that is dense and open in the image closure. This fact is often used in algebraic geometry. Perhaps even more important is the fact that if X is irreducible, then there is exists a dense, open subset of $\overline{f(X)}$ such that for all y in this subset, the fibre dimension $\dim\left(f^{-1}(y)\right)$ equals $\dim(X) - \dim\left(\overline{f(X)}\right)$. Moreover (and without assuming X to be irreducible), all the subsets of Y where the fibre dimension has some given value are constructible. This statement is also due to Chevalley.

The purpose of this paper is to turn all these statements into algorithms. We restrict to the case of affine varieties over an algebraically closed field, or, more generally, affine schemes of finite type over a field. The proofs of correctness of the algorithms also yield proofs of the above-mentioned theorems on constructible sets and fibre dimensions. As might be expected, the algorithms use Gröbner basis methods. But no computation of irreducible components is required.

The paper is organized as follows. In the first section we give a recursive algorithm that finds the image of a variety $X \subseteq K^{n+m}$ under the projection $K^{n+m} \to K^m$. This image is given as a constructible set. Applying this to the graph of a morphism immediately yields an algorithm for computing the image of a morphism of affine varieties. This extends in a straightforward way to the computation of images of constructible subsets. The partition of the image into sets of constant fibre dimension is a by-product. In Section 2 we give an algorithm for computing a subset of a constructible set S which is dense and open in $\overline{S}$. The first step is to write a constructible set as a disjoint union of locally closed sets, for which we also give an algorithm. In the final section, we show that the algorithms also work in the situation of affine schemes of finite type over a field.

Notation. Throughout the paper, K will be a field, which in the first two sections will be assumed to be algebraically closed. If $I \subseteq K[x_1, \ldots, x_n]$ is a subset of a polynomial ring, we write $\operatorname{Var}_{K^n}(I)$ for the affine variety given by I. Conversely, if $X \subseteq K^n$ is a subset, we write $\operatorname{Id}_{K[x_1,\ldots,x_n]}(X)$ for the vanishing ideal of X. We write $\overline{X} = \operatorname{Var}_{K^n}\left(\operatorname{Id}_{K[x_1,\ldots,x_n]}(X)\right)$ for the Zariski closure. We will often write $K[\underline{x}]$ for $K[x_1, \ldots, x_n]$ if the number of indeterminates is clear. For a subset $S \subseteq R$ of a commutative ring R, we write $\langle S \rangle_R$ for the ideal generated by S.

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1 Images of morphisms

In this section K is assumed to be algebraically closed. Let "$>_x$" and "$>_y$" be monomial orderings on polynomial rings $K[\underline{x}] = K[x_1, \ldots, x_n]$ and $K[\underline{y}] = K[y_1, \ldots, y_m]$, respectively, and let "$>$" be the block ordering (sometimes also called the *product ordering*) on $K[\underline{x}, \underline{y}]$ formed from "$>_x$" and "$>_y$" with $x_i > y_j$. We can view a polynomial $f \in K[\underline{x}, \underline{y}] \setminus \{0\}$ as a polynomial with coefficients in $K[\underline{y}]$ and indeterminates $x_1, \ldots, x_n$, and consider its leading monomial $\operatorname{LM}_x(f) \in K[\underline{x}]$ and leading coefficient $\operatorname{LC}_x(f) \in K[\underline{y}]$ w.r.t. "$>_x$". Likewise, for $f, g \in K[\underline{x}, \underline{y}] \setminus \{0\}$ we define the s-polynomial with respect to the x-variables as

$$\operatorname{spol}_x(f,g) := \frac{\operatorname{lcm}\left(\operatorname{LM}_x(f), \operatorname{LM}_x(g)\right)}{\operatorname{LM}_x(f)} \operatorname{LC}_x(g) \cdot f - \frac{\operatorname{lcm}\left(\operatorname{LM}_x(f), \operatorname{LM}_x(g)\right)}{\operatorname{LM}_x(g)} \operatorname{LC}_x(f) \cdot g \in K[\underline{x}, \underline{y}].$$

If $\mathcal{G} \subset K[\underline{x}, \underline{y}]$ is a Gröbner basis, we write $\operatorname{NF}_{\mathcal{G}}(f)$ for the normal form of f. Consider the projection

$$\pi \colon K^{n+m} \to K^m, \ (\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_m) \mapsto (\eta_1, \ldots, \eta_m).$$

The following algorithm provides the core of the main algorithms of this section (Algorithms 1.4 and 1.6).

Algorithm 1.1 (Partial image of a projection)**.**

Input: An ideal $I \subseteq K[\underline{x}, \underline{y}]$ defining a variety $X = \operatorname{Var}_{K^{n+m}}(I)$.

Output: An ideal $J \subseteq K[\underline{y}]$ and a finite set $M \subset K[\underline{y}]$ with $M \cap I = \emptyset$ such that with

$$Y := \operatorname{Var}_{K^m}(J) \quad \text{and} \quad Z := \operatorname{Var}_{K^m}\Bigl(\prod_{g \in M \cup \{1\}} g\Bigr)$$

we have

$$\overline{\pi(X)} = Y \tag{1.1}$$

and

$$\pi(X) \setminus Z = Y \setminus Z. \tag{1.2}$$

Optionally, the algorithm computes a non-negative integer d such that the fibre dimension is given by

$$\dim\left(X \cap \pi^{-1}(\eta_1, \ldots, \eta_m)\right) = d \quad \text{for all} \quad (\eta_1, \ldots, \eta_m) \in Y \setminus Z. \tag{1.3}$$

In particular, the fibre dimension is constant on $Y \setminus Z$.

(1) Compute a reduced Gröbner basis $\mathcal{G}$ of I w.r.t. "$>$".

(2) Set

$$\mathcal{G}_y := \mathcal{G} \cap K[\underline{y}], \quad \mathcal{G}_x := \mathcal{G} \setminus \mathcal{G}_y \quad \text{and} \quad J := \langle \mathcal{G}_y \rangle_{K[\underline{y}]}.$$

(3) Compute

$$S := \left\{ \mathrm{NF}_{\mathcal{G}_y}\left(\mathrm{spol}_x(f, f')\right) \mid f, f' \in \mathcal{G}_x \right\} \setminus \{0\}$$

and

$$M := \{\mathrm{LC}_x(s) \mid s \in S\} \cup \{\mathrm{LC}_x(f) \mid f \in \mathcal{G}_x\}.$$

(4) Optionally, compute

$$d := \dim\left(K[\underline{x}] \Big/ \langle \mathrm{LM}_x(f) \mid f \in \mathcal{G}_x \rangle_{K[\underline{x}]} \right).$$

The dimension of the monomial ideal can be computed by using Cox et al. [2, Proposition 3 of Chapter 9, § 1].

Proof of correctness of Algorithm 1.1. By the nature of the ordering "$>$", we have that $\mathcal{G}_y$ is a Gröbner basis of the elimination ideal $I \cap K[\underline{y}]$ (see Becker and Weispfenning [1, Proposition 6.15]), so $J = I \cap K[\underline{y}]$. Since $\mathrm{Id}_{K[\underline{y}]}(\pi(X)) = \sqrt{I} \cap K[\underline{y}]$, Equation (1.1) follows.

For a polynomial $h \in K[\underline{x}, \underline{y}]$ with $\mathrm{NF}_{\mathcal{G}_y}(h) \neq 0$ we have $\mathrm{LC}_x\left(\mathrm{NF}_{\mathcal{G}_y}(h)\right) \notin J$, so $\mathrm{LC}_x\left(\mathrm{NF}_{\mathcal{G}_y}(h)\right) \notin I$. Thus the $\mathrm{LC}_x\left(\mathrm{NF}_{\mathcal{G}_y}\left(\mathrm{spol}_x(f, f')\right)\right)$ appearing in M do not lie in I. Neither do the $\mathrm{LC}_x(f)$ with $f \in \mathcal{G}_x$, since $f = \mathrm{NF}_{\mathcal{G}_y}(f)$ follows from the reducedness of $\mathcal{G}$. Thus $M \cap I = \emptyset$.

The inclusion "$\subseteq$" of (1.2) follows from (1.1). To prove the reverse inclusion, take $(\eta_1, \ldots, \eta_m) \in Y \setminus Z$. Consider the specialization homomorphism φ: $K[\underline{x}, \underline{y}] \to K[\underline{x}]$ sending y_i to η_i. Since $X = \mathrm{Var}_{K^{n+m}}(\mathcal{G})$, it follows that $\{(\xi_1, \ldots, \xi_n) \in K^n \mid (\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_m) \in X\} = \mathrm{Var}_{K^n}(\varphi(\mathcal{G}))$. Since $(\eta_1, \ldots, \eta_m) \in Y$, we get $\varphi(\mathcal{G}_y) \subseteq \{0\}$, so

$$\{(\xi_1, \ldots, \xi_n) \in K^n \mid (\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_m) \in X\} = \mathrm{Var}_{K^n}\left(\varphi(\mathcal{G}_x)\right). \tag{1.4}$$

We claim that $\varphi(\mathcal{G}_x)$ is a Gröbner basis w.r.t. "$>_x$". Take $f, f' \in \mathcal{G}_x$. We need to show that the s-polynomial $\mathrm{spol}\left(\varphi(f), \varphi(f')\right)$ has normal form 0 w.r.t. $\varphi(\mathcal{G}_x)$. We may assume $\mathrm{spol}\left(\varphi(f), \varphi(f')\right) \neq 0$. Since $\mathrm{LC}_x(f) \in M$ and $(\eta_1, \ldots, \eta_m) \notin Z$ we have $\varphi\left(\mathrm{LC}_x(f)\right) \neq 0$, and likewise $\varphi\left(\mathrm{LC}_x(f')\right) \neq 0$. This implies

$$\mathrm{LM}_x\left(\varphi(f)\right) = \mathrm{LM}_x(f) \quad \text{and} \quad \mathrm{LC}_x\left(\varphi(f)\right) = \varphi\left(\mathrm{LC}_x(f)\right), \tag{1.5}$$

and the same for f'. Thus

$$\mathrm{spol}\left(\varphi(f), \varphi(f')\right) = \varphi\left(\mathrm{spol}_x(f, f')\right).$$

Setting

$$s := \mathrm{NF}_{\mathcal{G}_y}\left(\mathrm{spol}_x(f, f')\right)$$

we obtain

$$\mathrm{spol}\left(\varphi(f), \varphi(f')\right) = \varphi(s).$$

In particular, $s \neq 0$. Since s lies in I, the Gröbner basis property of $\mathcal{G}$ yields have a representation

$$s = \sum_{g \in \mathcal{G}} h_g \cdot g \quad \text{with} \quad h_g \in K[\underline{x}, \underline{y}] \quad \text{and} \quad \mathrm{LM}_>(h_g \cdot g) \leqslant \mathrm{LM}_>(s) \quad \text{for} \quad h_g \neq 0.$$

Thus

$$\varphi(s) = \sum_{g \in \mathcal{G}_x} \varphi(h_g) \cdot \varphi(g), \tag{1.6}$$

where it suffices to take the summands with $g \in \mathcal{G}_x$ since $(\eta_1, \ldots, \eta_n) \in Y$. Therefore (1.6) yields the desired representation for $\mathrm{spol}\left(\varphi(f), \varphi(f')\right)$ if we can show that $\mathrm{LM}_x\left(\varphi(h_g) \cdot \varphi(g)\right) \leqslant \mathrm{LM}_x\left(\varphi(s)\right)$ for all $g \in \mathcal{G}_x$ with $\varphi(h_g) \neq 0$. By the nature of the block ordering we have for any $h \in K[\underline{x}, \underline{y}] \setminus \{0\}$:

$$\mathrm{LM}_>(h) = \mathrm{LM}_x(h) \cdot \mathrm{LM}_{>_y}\left(\mathrm{LC}_x(h)\right), \quad \text{so} \quad \mathrm{LM}_x(h) = \mathrm{LM}_>(h)|_{y_1 = \cdots = y_m = 1}.$$

Thus

$$\mathrm{LM}_x\left(\varphi(h_g \cdot g)\right) \leqslant \mathrm{LM}_x(h_g \cdot g) = \mathrm{LM}_>(h_g \cdot g)\big|_{y_i=1} \leqslant \mathrm{LM}_>(s)\big|_{y_i=1} = \mathrm{LM}_x(s) = \mathrm{LM}_x\left(\varphi(s)\right),$$

where the second inequality follows since $\mathrm{LM}_>(h_g \cdot g) \leqslant \mathrm{LM}_>(s)$, and the last equality follows since $\mathrm{LC}_x(s) \in M$ and $(\eta_1, \ldots, \eta_m) \notin Z$. This completes the proof that $\varphi(\mathcal{G}_x)$ is a Gröbner basis.

But $\varphi(\mathcal{G}_x)$ contains no constant since (1.5) implies that $\mathrm{LM}_x\left(\varphi(f)\right)$ involves some x-variables for all $f \in \mathcal{G}_x$. Thus $\mathrm{Var}_{K^n}\left(\varphi(\mathcal{G}_x)\right) \neq \emptyset$ by the Nullstellensatz, so by (1.4) we obtain that $(\eta_1, \ldots, \eta_m) \in \pi(X)$. This completes the proof of (1.2).

To show the correctness of step 4 we observe that the variety $X \cap \pi^{-1}(\eta_1, \ldots, \eta_m)$ is isomorphic to $\{(\xi_1, \ldots, \xi_n) \in K^n \mid (\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_m) \in X\}$, so by (1.4) its dimension equals the (Krull) dimension of $K[\underline{x}] \big/ \langle \varphi(\mathcal{G}_x) \rangle_{K[\underline{x}]}$. By Greuel and Pfister [3, Corollary 7.5.5], the dimension of an ideal is equal to the dimension of its initial ideal. Since $\varphi(\mathcal{G}_x)$ is a Gröbner basis, the initial ideal of $\langle \varphi(\mathcal{G}_x) \rangle_{K[\underline{x}]}$ is generated by the $\mathrm{LM}_x\left(\varphi(f)\right)$, $f \in \mathcal{G}_x$. Thus by (1.5) we obtain that

$$\dim\left(K[\underline{x}] \big/ \langle \varphi(\mathcal{G}_x) \rangle_{K[\underline{x}]}\right) = \dim\left(K[\underline{x}] \big/ \langle \mathrm{LM}_x(f) \mid f \in \mathcal{G}_x \rangle_{K[\underline{x}]}\right).$$

Therefore step 4 of the algorithm determines the dimension of $X \cap \pi^{-1}(\eta_1, \ldots, \eta_m)$ correctly. □

Remark 1.2. (a) Suppose the ideal I from Algorithm 1.1 is a radical ideal and $1 \notin I$. Then the same is true for $J = I \cap K[\underline{y}]$, so $Y \setminus Z$ is non-empty. If I is even a prime ideal, then $Y \setminus Z$ is a subset of $\pi(X)$ which is open and dense in $Y = \overline{\pi(X)}$.

If I is not a prime ideal, Algorithms 1.4 and 2.4 can be used to find a subset of $\pi(X)$ which is open and dense in $\overline{\pi(X)}$.

(b) If I is a prime ideal, then the dimensions of X and Y are the transcendence degree of $K[\underline{x}, \underline{y}]/I$ and $K[\underline{y}]/J$, respectively. It follows immediately that in Algorithm 1.1 we have

$$d = \dim(X) - \dim(Y).$$

So we get a proof of the fact that on a dense, open subset the fibre dimension equals the codimension $\dim(X) - \dim(Y)$ (see Hartshorne [4, Exercise 3.22(c)]).

If I is not a prime ideal, it follows from the way in which the dimension is computed from the leading monomials of a Gröbner basis that

$$d \leqslant \dim(X) - \dim(Y).$$

The inequality may be strict, as the example $I = \langle x_1 y, x_2 y \rangle \subset K[x_1, x_2, y]$ shows. ◁

We use a recursive call to take care of the variety Z that is "left behind" by Algorithm 1.1. This enables us to compute the full image of a projection.

Algorithm 1.3 (Image of a projection)**.**

Input: An ideal $I \subseteq K[\underline{x}, \underline{y}]$ defining an affine variety $X = \mathrm{Var}_{K^{n+m}}(I)$.

Output: Ideals $J_1, \ldots, J_l \subseteq K[\underline{y}]$ and polynomials $g_1, \ldots, g_l \in K[\underline{y}]$ such that with

$$Y_i := \mathrm{Var}_{K^m}(J_i) \quad \text{and} \quad Z_i := \mathrm{Var}_{K^m}(g_i)$$

we have

$$\pi(X) = \bigcup_{i=1}^{l} (Y_i \setminus Z_i). \tag{1.7}$$

and

$$\overline{\pi(X)} = Y_1. \tag{1.8}$$

Optionally, the algorithm also computes non-negative integers $d_1, \ldots, d_l$ such that the fibre dimensions are given by

$$\dim\left(X \cap \pi^{-1}(\eta_1, \ldots, \eta_m)\right) = d_i \quad \text{for all} \quad (\eta_1, \ldots, \eta_m) \in Y_i \setminus Z_i. \tag{1.9}$$

(1) Choose monomial orderings "$>_x$" and "$>_y$" on $K[\underline{x}]$ and $K[\underline{y}]$ and let "$>$" be the block ordering on $K[\underline{x},\underline{y}]$ formed from "$>_x$" and "$>_y$" with $x_i > y_j$.

(2) Let J, M and optionally d be the result of applying Algorithm 1.1 to I. Initialize lists of J_i, g_i and (optimally) d_i by

$$J_1 := J, \quad g_1 := \prod_{g \in M \cup \{1\}} g \quad \text{and (optionally)} \quad d_1 = d.$$

(One may also choose g_1 to be a common multiple of all $g \in M$ dividing the product.)

(3) For all $g \in M$ perform step 4.

(4) Apply Algorithm 1.3 recursively to $\langle I \cup \{g\}\rangle_{K[\underline{x},\underline{y}]}$ and append the resulting J_i, g_i and (optionally) d_i to the current lists.

Proof of correctness of Algorithm 1.3. Since $g \notin I$ for each $g \in M$, the ideal I is replaced by a strictly larger ideal in each recursion level. Hence termination of the algorithm follows by Noetherian induction.

Also by induction we may assume that for each $g \in M$ step 4 yields closed subsets $Y_{g,i}, Z_{g,i} \subseteq K^m$ and (optionally) $d_{g,i} \in \mathbb{N}_0$ $(i = 1, \ldots, l_g)$ such that

$$\pi(X) \cap \mathrm{Var}_{K^m}(g) = \pi\left(\mathrm{Var}_{K^{n+m}}(I \cup \{g\})\right) = \bigcup_{i=1}^{l_g} (Y_{g,i} \setminus Z_{g,i}) \tag{1.10}$$

and

$$\dim\left(\mathrm{Var}_{K^{n+m}}(I \cup \{g\}) \cap \pi^{-1}(\eta_1, \ldots, \eta_m)\right) = d_{g,i} \quad \text{for all} \quad (\eta_1, \ldots, \eta_m) \in Y_{g,i} \setminus Z_{g,i}. \tag{1.11}$$

Let $(\eta_1, \ldots, \eta_m) \in Y_{g,i} \setminus Z_{g,i}$. Then (1.10) implies $g(\eta_1, \ldots, \eta_m) = 0$, so

$$X \cap \pi^{-1}(\eta_1, \ldots, \eta_m) = \mathrm{Var}_{K^{n+m}}(I \cup \{g\}) \cap \pi^{-1}(\eta_1, \ldots, \eta_m).$$

Therefore (1.11) implies that (1.9) is satisfied. Putting (1.2) and (1.10) together yields

$$\pi(X) = \pi(X) \cap \Big((K^m \setminus \mathrm{Var}_{K^m}(g_1)) \cup \bigcup_{g \in M} \mathrm{Var}_{K^m}(g) \Big) = (Y_1 \setminus Z_1) \cup \bigcup_{g \in M} \bigcup_{i=1}^{l_g} (Y_{g,i} \setminus Z_{g,i}),$$

which is (1.7). Equation (1.8) follows from (1.1). □

Remark. The case where X is a closed subset of $\mathbb{P}^n \times \mathbb{A}^n$ and $\pi\colon \mathbb{P}^n \times \mathbb{A}^n \to \mathbb{A}^n$ is the canonical projection is treated by Greuel and Pfister [3, Section A7]. In this case, $\pi(X)$ is always closed in $\mathbb{A}^m$. ◁

Recall that a set $Y \subseteq K^m$ is called *constructible* if it can be written as a finite union of sets of the form $Y_i \setminus Z_i$ with $Y_i, Z_i \subseteq K^m$ closed (see Hartshorne [4, Exercise 3.18]).

Algorithm 1.4 (Image of an affine variety as a constructible set)**.**

Input: An ideal $I \subseteq K[x_1, \ldots, x_n]$ defining an affine variety $X = \mathrm{Var}_{K^n}(I)$, and polynomials $f_1, \ldots, f_m \in K[x_1, \ldots, x_n]$ defining a morphism $f\colon X \to K^m$.

Output: Ideals $J_1, \ldots, J_l \subseteq K[y_1, \ldots, y_m]$ and polynomials $g_1, \ldots, g_l \in K[y_1, \ldots, y_m]$ such that with

$$Y_i := \mathrm{Var}_{K^m}(J_i) \quad \text{and} \quad Z_i := \mathrm{Var}_{K^m}(g_i)$$

we have

$$f(X) = \bigcup_{i=1}^{l} (Y_i \setminus Z_i).$$

and

$$\overline{f(X)} = Y_1.$$

Optionally, the algorithm also computes non-negative integers $d_1, \dots, d_l$ such that the fibre dimensions are given by

$$\dim\left(f^{-1}(\eta_1, \dots, \eta_m)\right) = d_i \quad \text{for all} \quad (\eta_1, \dots, \eta_m) \in Y_i \setminus Z_i.$$

(1) Form the ideal

$$J := \langle I \cup \{f_1 - y_1, \dots, f_m - y_m\}\rangle_{K[\underline{x},\underline{y}]} \subseteq K[x_1, \dots, x_n, y_1, \dots, y_m].$$

(2) Apply Algorithm 1.3 to J.

Proof of correctness of Algorithm 1.4. Consider the graph

$$\Gamma := \left\{(\xi_1, \dots, \xi_n, \eta_1, \dots, \eta_m) \in K^{n+m} \mid (\xi_1, \dots, \xi_n) \in X,\ (\eta_1, \dots, \eta_m) = f(\xi_1, \dots, \xi_n)\right\}$$

of f. We have $\Gamma = \mathrm{Var}_{K^{n+m}}(J)$. Moreover, with

$$\pi\colon K^{n+m} \to K^m,\ (\xi_1, \dots, \xi_n, \eta_1, \dots, \eta_m) \mapsto (\eta_1, \dots, \eta_m)$$

we have $f(X) = \pi(\Gamma)$. So it follows from the correctness of Algorithm 1.3 that Algorithm 1.4 computes $f(X)$ and $\overline{f(X)}$ correctly. Algorithm 1.3 (optionally) computes the dimensions of $\Gamma \cap \pi^{-1}(y_1, \dots, y_m)$ for $(\eta_1, \dots, \eta_m) \in f(X)$. But projection on the first n coordinates provides an isomorphism from $\Gamma \cap \pi^{-1}(\eta_1, \dots, \eta_m)$ to the fibre $f^{-1}(\eta_1, \dots, \eta_m)$. Thus Algorithm 1.4 computes the fibre dimensions correctly, too. □

Remark 1.5. (a) Algorithm 1.4 provides a constructive version of the following theorem by Chevalley (see Hartshorne [4, Exercise 3.22(e)]): For a morphism $f\colon X \to Y$ of affine varieties and for a non-negative integer d the set

$$C_d := \left\{y \in Y \mid \dim\left(f^{-1}(y)\right) = d\right\}$$

is constructible.

(b) If the ideal I in Algorithm 1.4 is a prime ideal, then Algorithm 1.1 applied to J produces a subset $Y \setminus Z$ of $f(X)$ which is dense and open in $\overline{f(X)}$. In fact, in this case J is also a prime ideal since $K[x_1, \dots, x_n, y_1, \dots, y_m]/J \cong K[x_1, \dots, x_n]/I$, so Remark 1.2(a) applies. From Remark 1.2(b), we see that $d = \dim(X) - \dim\left(\overline{f(X)}\right)$. ◁

We extend Algorithm 1.4 further and obtain an algorithm for computing the image of a constructible set.

Algorithm 1.6 (Image of a constructible set)**.**

Input: A constructible set $X \subseteq K^n$ given as

$$X = \bigcup_{i=1}^{r} \left(\mathrm{Var}_{K^n}(I_i) \setminus \mathrm{Var}_{K^n}(A_i)\right)$$

with $I_i, A_i \subseteq K[x_1, \dots, x_n]$ ideals, and a morphism $f\colon K^n \to K^m$ given by polynomials $f_1, \dots, f_m \in K[x_1, \dots, x_n]$.

Output: Ideals $J_1, \dots, J_l \subseteq K[y_1, \dots, y_m]$ and polynomials $g_1, \dots, g_l \in K[y_1, \dots, y_m]$ such that with

$$Y_i := \mathrm{Var}_{K^m}(J_i) \quad \text{and} \quad Z_i := \mathrm{Var}_{K^m}(g_i)$$

we have

$$f(X) = \bigcup_{i=1}^{l} \left(Y_i \setminus Z_i\right).$$

(1) If $A_i = \langle h_{i,1}, \ldots, h_{i,s_i} \rangle_{K[\underline{x}]}$, write

$$\operatorname{Var}_{K^n}(I_i) \setminus \operatorname{Var}_{K^n}(A_i) = \bigcup_{j=1}^{s_i} \left(\operatorname{Var}_{K^n}(I_i) \setminus \operatorname{Var}_{K^n}(h_{i,j})\right).$$

Doing this, we may assume that every A_i is of the form $A_i = \langle h_i \rangle_{K[\underline{x}]}$.

(2) Introduce a new indeterminate x_0 and let $\bar{f}\colon K^{n+1} \to K^m$ be the morphism given by $f_1, \ldots, f_m \in K[x_1, \ldots, x_n] \subset K[x_0, \ldots, x_n]$.

(3) For $i \in \{1, \ldots, r\}$, set

$$X_i := \operatorname{Var}_{K^{n+1}}\left(I_i \cup \{x_0 h_i - 1\}\right)$$

and apply Algorithm 1.4 to obtain closed subsets $Y_{i,j}, Z_{i,j} \subseteq K^m$ with

$$\bar{f}\left(X_i\right) = \bigcup_{j=1}^{l_i} \left(Y_{i,j} \setminus Z_{i,j}\right).$$

(4) We have

$$f(X) = \bigcup_{i=1}^{r} \bigcup_{j=1}^{l_i} \left(Y_{i,j} \setminus Z_{i,j}\right).$$

Proof of correctness of Algorithm 1.6. The reduction in step 1 is clear. Consider the projection

$$\pi\colon K^{n+1} \to K^n,\ (\xi_0, \ldots, \xi_n) \mapsto (\xi_1, \ldots, \xi_n).$$

We have $\bar{f} = f \circ \pi$ and

$$\pi\left(X_i\right) = \operatorname{Var}_{K^n}(I_i) \setminus \operatorname{Var}_{K^n}(h_i),$$

so

$$f\left(\operatorname{Var}_{K^n}(I_i) \setminus \operatorname{Var}_{K^n}(h_i)\right) = \bar{f}\left(X_i\right) = \bigcup_{j=1}^{l_i} \left(Y_{i,j} \setminus Z_{i,j}\right).$$

Now the correctness follows since

$$f(X) = \bigcup_{i=1}^{r} f\left(\operatorname{Var}_{K^n}(I_i) \setminus \operatorname{Var}_{K^n}(h_i)\right).$$

□

Remark 1.7. (a) Algorithm 1.6 provides a constructive version of the following theorem by Chevalley (see Hartshorne [4, Exercise 3.19]): A morphism $f\colon X \to Y$ of affine varieties maps constructible subsets of X to constructible subsets of Y.

(b) It is also possible, though a bit messy, to use Algorithm 1.6 for computing dimensions of fibres $X \cap f^{-1}(\eta_1, \ldots, \eta_m)$. The difficulty is that fibres may be composed of different parts coming from different pieces of X. In fact, for $(\eta_1, \ldots, \eta_m) \in f(X)$, we have

$$X \cap f^{-1}(\eta_1, \ldots, \eta_m) = \bigcup_{i=1}^{r} \left(\pi(X_i) \cap f^{-1}(\eta_1, \ldots, \eta_m)\right)$$

with π the projection on the last n components, so

$$\dim\left(X \cap f^{-1}(\eta_1, \ldots, \eta_m)\right) = \max\left\{ \dim\left(X_i \cap \bar{f}^{-1}(\eta_1, \ldots, \eta_m)\right) \,\middle|\, i = 1, \ldots, r \right\}.$$

By having Algorithm 1.4 compute the fibre dimensions, we obtain non-negative integers $d_{i,j}$ such that

$$\dim\left(X_i \cap \bar{f}^{-1}(\eta_1,\dots,\eta_m)\right) = d_{i,j} \quad \text{for all} \quad (\eta_1,\dots,\eta_m) \in Y_{i,j} \setminus Z_{i,j},$$

so

$$\dim\left(X \cap f^{-1}(\eta_1,\dots,\eta_m)\right) = \max\left\{d_{i,j} \mid (\eta_1,\dots,\eta_m) \in Y_{i,j} \setminus Z_{i,j}\right\}.$$

So one can do the following: Apply Algorithm 2.2 (appearing in the next section) to $f(X) = \bigcup_{i=1}^r \bigcup_{j=1}^{l_i} (Y_{i,j} \setminus Z_{i,j})$, but feed the sets $Y_{i,j} \setminus Z_{i,j}$ into the algorithm ordered by decreasing fibre dimensions $d_{i,j}$. This yields the set $f(X)$ as a disjoint union, which by Remark 2.3(a) has the property that every subset S from this union lies in some $Y_{i,j} \setminus Z_{i,j}$, but does not intersect with any of those sets $Y_{i',j'} \setminus Z_{i',j'}$ having $d_{i',j'} > d_{i,j}$. So all points from S have fibre dimension $d_{i,j}$. This procedure determines all

$$C_d := \left\{y \in K^m \mid \dim\left(X \cap f^{-1}(y)\right) = d\right\},$$

and shows that they are constructive, which is a slight generalization of Remark 1.5(a). $\triangleleft$

2 Dense open subsets

During this section we continue to assume that K is algebraically closed. The goal of this section is to explicitly find a subset of a constructible set X which is open and dense in the closure $\overline{X}$. We start by the following algorithm.

Algorithm 2.1 (Closure of a locally closed set)**.**

Input: Ideals $I, A \subseteq K[x_1,\dots,x_n]$ defining a set $X = \mathrm{Var}_{K^n}(I) \setminus \mathrm{Var}_{K^n}(A)$.

Output: An ideal $J \subseteq K[x_1,\dots,x_n]$ such that

$$\overline{X} = \mathrm{Var}_{K^n}(J).$$

(1) Suppose $A = \langle h_1,\dots,h_r\rangle_{K[x_1,\dots,x_n]}$. With an additional indeterminate x_0, set

$$g := \prod_{i=1}^r (x_0 h_i - 1).$$

(2) Compute the elimination ideal

$$J := \langle I \cup \{g\}\rangle_{K[x_0,\dots,x_n]} \cap K[x_1,\dots,x_n].$$

Proof of correctness of Algorithm 2.1. With

$$\pi\colon K^{n+1} \to K^n,\ (\xi_0,\dots,\xi_n) \mapsto (\xi_1,\dots,\xi_n) \quad \text{and} \quad Y = \mathrm{Var}_{K^{n+1}}(I \cup \{g\})$$

we have $\mathrm{Var}_{K^n}(J) = \overline{\pi(Y)}$, so it suffices to show that $\pi(Y) = X$. But this is clear. $\square$

Algorithm 2.2 (Write a constructible set as a disjoint union)**.**

Input: A constructible subset $X \subseteq K^n$ given as

$$X = \bigcup_{i=1}^r \left(\mathrm{Var}_{K^n}(I_i) \setminus \mathrm{Var}_{K^n}(A_i)\right)$$

with $I_i, A_i \subseteq K[x_1,\dots,x_n]$ ideals.

Output: A finite set M consisting of pairs (J, B) with $J, B \subseteq K[x_1, \ldots, x_n]$ ideals, such that

$$X = \dot{\bigcup_{(J,B)\in M}} \left(\mathrm{Var}_{K^n}(J) \setminus \mathrm{Var}_{K^n}(B)\right)$$

(disjoint union). Optionally, the algorithm also achieves that all $(J, B) \in M$ satisfy

$$\mathrm{Var}_{K^n}(J) = \overline{\mathrm{Var}_{K^n}(J) \setminus \mathrm{Var}_{K^n}(B)} \tag{2.1}$$

and

$$\mathrm{Var}_{K^n}(J) \neq \emptyset. \tag{2.2}$$

(1) Set $M := \emptyset$ and $N := \left\{\left(\{0\}, \langle 1\rangle_{K[\underline{x}]}\right)\right\}$. (Like M, N will be a set of pairs of ideals.)

(2) For $i \in \{1, \ldots, r\}$ perform the steps 3 – 7.

(3) Set

$$M := M \cup \{(I_i + J, A_i \cap B) \mid (J, B) \in N\}.$$

(4) Set

$$N := \bigcup_{(J,B)\in N} \{(J, I_i \cap B), (I_i + A_i + J, B)\}.$$

(5) (Optional) If (2.1) should be satisfied, apply Algorithm 2.1 to all $(J, B) \in M$ and substitute each J by the result of Algorithm 2.1.

(6) (Optional) If (2.2) should be satisfied, delete all pairs (J, B) from M where J contains a non-zero constant.

(7) (Optional) Perform steps 5 and/or 6 on N. (This step may speed up the computation.)

Remark. The intersections of ideals appearing in steps 3 and 4 of Algorithm 2.2 may be replaced by ideal products. ◁

Proof of correctness of Algorithm 2.2. Let M_i and N_i be the sets M and N after the i-th passage through the loop. Moreover, set

$$X_i := \bigcup_{j=1}^{i} \left(\mathrm{Var}(I_j) \setminus \mathrm{Var}(A_j)\right).$$

We claim that

$$X_i = \dot{\bigcup_{(J,B)\in M_i}} \left(\mathrm{Var}(J) \setminus \mathrm{Var}(B)\right) \tag{2.3}$$

and

$$X_i^c := K^n \setminus X_i = \dot{\bigcup_{(J,B)\in N_i}} \left(\mathrm{Var}(J) \setminus \mathrm{Var}(B)\right) \tag{2.4}$$

(disjoint unions). This is true for $i = 0$ by the initialization in step 1. Assume $i > 0$ and use induction. Then

$$X_i = X_{i-1} \cup \left(\mathrm{Var}(I_i) \setminus \mathrm{Var}(A_i)\right) =$$

$$\Bigg(\dot{\bigcup_{(J,B)\in M_{i-1}}} \left(\mathrm{Var}(J) \setminus \mathrm{Var}(B)\right)\Bigg) \,\dot\cup\, \Big(X_{i-1}^c \cap \left(\mathrm{Var}(I_i) \setminus \mathrm{Var}(A_i)\right)\Big) =$$

$$\Bigg(\dot{\bigcup_{(J,B)\in M_{i-1}}} \left(\mathrm{Var}(J) \setminus \mathrm{Var}(B)\right)\Bigg) \,\dot\cup\, \dot{\bigcup_{(J,B)\in N_{i-1}}} \Big(\left(\mathrm{Var}(J) \setminus \mathrm{Var}(B)\right) \cap \left(\mathrm{Var}(I_i) \setminus \mathrm{Var}(A_i)\right)\Big).$$

But we have

$$(\mathrm{Var}(J) \setminus \mathrm{Var}(B)) \cap (\mathrm{Var}(I_i) \setminus \mathrm{Var}(A_i)) = \mathrm{Var}(I_i + J) \setminus \mathrm{Var}(A_i \cap B),$$

so step 3 provides that (2.3) is satisfied for i. Moreover,

$$X_i^c = X_{i-1}^c \cap (\mathrm{Var}(I_i)^c \cup \mathrm{Var}(A_i)) = \\ \left(\dot{\bigcup_{(J,B)\in N_{i-1}}} (\mathrm{Var}(J) \setminus \mathrm{Var}(B)) \right) \cap \left(\mathrm{Var}(I_i)^c \,\dot{\cup}\, (\mathrm{Var}(A_i) \cap \mathrm{Var}(I_i)) \right).$$

Thus we get X_i^c as a disjoint union of all

$$\begin{aligned} (\mathrm{Var}(J) \setminus \mathrm{Var}(B)) \cap \mathrm{Var}(I_i)^c &= \mathrm{Var}(J) \setminus \mathrm{Var}(I_i \cap B), \\ (\mathrm{Var}(J) \setminus \mathrm{Var}(B)) \cap (\mathrm{Var}(A_i) \cap \mathrm{Var}(I_i)) &= \mathrm{Var}(I_i + A_i + J) \setminus \mathrm{Var}(B), \end{aligned}$$

where (J, B) runs through N. So step 4 provides that (2.4) is satisfied for i.

The correctness of the optional steps is clear. □

Remark 2.3. (a) From (2.3) in the proof, we get the following additional property of the partition obtained in Algorithm 2.2. If M_i is the set M formed the i-th passage of step 3, then

$$\dot{\bigcup_{(J,B)\in M_i \setminus M_{i-1}}} (\mathrm{Var}(J) \setminus \mathrm{Var}(B)) = (\mathrm{Var}(I_i) \setminus \mathrm{Var}(A_i)) \setminus \left(\bigcup_{j=1}^{i-1} (\mathrm{Var}(I_j) \setminus \mathrm{Var}(A_j)) \right).$$

(b) Algorithm 2.2 really works in any topological space where intersections and unions of closed sets can be computed explicitly. ◁

Algorithm 2.4 (Dense and open subset of a constructible set)**.**

Input: A constructible subset $X \subseteq K^n$ given as

$$X = \bigcup_{i=1}^{r} (\mathrm{Var}_{K^n}(I_i) \setminus \mathrm{Var}_{K^n}(A_i)) \tag{2.5}$$

with $I_i, A_i \subseteq K[x_1, \ldots, x_n]$ ideals.

Output: Ideals $I, A \subseteq K[x_1, \ldots, x_n]$ such that $\mathrm{Var}_{K^n}(I) = \overline{X}$, and with $U := \mathrm{Var}_{K^n}(I) \setminus \mathrm{Var}_{K^n}(A)$ we have

$$U \subseteq X \quad \text{and} \quad \overline{U} = \overline{X}.$$

(1) Apply Algorithm 2.2 to X. We will now assume that the union in (2.5) is disjoint and

$$\mathrm{Var}_{K^n}(I_i) = \overline{\mathrm{Var}_{K^n}(I_i) \setminus \mathrm{Var}_{K^n}(A_i)}$$

holds for all i.

(2) Compute $I := \bigcap_{i=1}^r I_i$.

(3) For $i \in \{1, \ldots, r\}$ compute $B_i := A_i \cap \bigcap_{j \neq i} I_j$.

(4) Set $A := B_1 + \cdots + B_r$.

Remark. The intersections of ideals appearing in steps 2 and 3 of Algorithm 2.4 may be replaced by ideal products. ◁

We need the following lemma for the proof of correctness of Algorithm 2.4.

Lemma 2.5. *Let $Y \subseteq Z$ be an irreducible component of an affine variety $Z \subseteq K^n$.*

(a) Let $U \subseteq Z$ be a subset with $Y \subseteq \overline{U}$. Then $Y \cap U \neq \emptyset$.

(b) Let $A, B \subseteq K^n$ be closed sets such that both $Y \setminus A$ and $Y \setminus B$ are non-empty. Then the intersection $(Y \setminus A) \cap (Y \setminus B)$ is non-empty, too.

(c) Let $U \subseteq Z$ be open with $U \cap Y \neq \emptyset$. Then $Y \subseteq \overline{U}$.

Proof. (a) Let $Y = Y_1, Y_2, \dots, Y_r$ be the irreducible components of Z. Then $U = \bigcup_{i=1}^r (U \cap Y_i)$ and $\overline{U} = \bigcup_{i=1}^r \overline{U \cap Y_i}$. Assume $U \cap Y = \emptyset$. Then $\overline{U} = \bigcup_{i=2}^r \overline{U \cap Y_i}$, so by hypothesis $Y_1 \subseteq \bigcup_{i=2}^r Y_i$, contradicting the irredundancy of the decomposition $Z = \bigcup_{i=1}^r Y_i$.

(b) Assume $(Y \setminus A) \cap (Y \setminus B) = \emptyset$. Then $Y = (Y \cap A) \cup (Y \cap B)$, so $Y \subseteq A$ or $Y \subseteq B$ by the irreducibility of Y, contradicting the hypothesis.

(c) We have $Y = \left(\overline{U} \cap Y\right) \cup \left(Y \setminus U\right)$. Since $Y \setminus U \subsetneqq Y$ and Y is irreducible, $Y \subseteq \overline{U}$ follows. □

Proof of correctness of Algorithm 2.4. Set $X_i := \mathrm{Var}(I_i) \setminus \mathrm{Var}(A_i)$. Then

$$\overline{X} = \bigcup_{i=1}^r \overline{X_i} = \bigcup_{i=1}^r \mathrm{Var}(I_i) = \mathrm{Var}(I).$$

Moreover, set $U_i := \overline{X} \setminus \mathrm{Var}(B_i)$. Then $U = \bigcup_{i=1}^r U_i$. Since $\mathrm{Var}(B_i) = \mathrm{Var}(A_i) \cup \bigcup_{j \neq i} \overline{X_j}$, we have

$$U_i \subseteq \overline{X_i} \setminus \mathrm{Var}(A_i) = X_i \subseteq X,$$

so $U \subseteq X$.

Let $Y \subseteq \overline{X}$ be an irreducible component of $\overline{X}$. We have $Y = \bigcup_{i=1}^r \left(Y \cap \overline{X_i}\right)$, so there exists an i with $Y \subseteq \overline{X_i}$. Thus $Y \cap X_i \neq \emptyset$ by Lemma 2.5(a). Let $j \in \{1, \dots, r\}$ be another index with $Y \subseteq \overline{X_j}$. Then also $Y \cap X_j \neq \emptyset$. Thus $Y \setminus \mathrm{Var}(A_i) \neq \emptyset$ and $Y \setminus \mathrm{Var}(A_j) \neq \emptyset$, so by Lemma 2.5(b) also $(Y \setminus \mathrm{Var}(A_i)) \cap (Y \setminus \mathrm{Var}(A_j)) \neq \emptyset$. But

$$(Y \setminus \mathrm{Var}(A_i)) \cap (Y \setminus \mathrm{Var}(A_j)) \subseteq \left(\overline{X_i} \setminus \mathrm{Var}(A_i)\right) \cap \left(\overline{X_j} \setminus \mathrm{Var}(A_j)\right) = X_i \cap X_j,$$

so $i = j$ by the disjointness of the union (2.5). Thus for $j \neq i$ we have $Y \not\subseteq \overline{X_j}$. It follows that

$$Y \cap U_i = Y \cap \left(\overline{X} \setminus (\mathrm{Var}(A_i) \cup \bigcup_{j \neq i} \overline{X_j})\right) = (Y \setminus \mathrm{Var}(A_i)) \cap \bigcap_{j \neq i} \left(Y \setminus \overline{X_j}\right) \neq \emptyset,$$

where the inequality follows by Lemma 2.5(b). WithLemma 2.5(c) we conclude that $Y \subseteq \overline{U_i}$, so $Y \subseteq \overline{U}$. Since this holds for every irreducible component Y of $\overline{X}$ we have $\overline{U} = \overline{X}$. This concludes the proof. □

Algorithm 2.4 yields a constructive proof of the following theorem, which is also due to Chevalley.

Theorem 2.6. *Every constructible set $X \subseteq K^n$ contains a subset $U \subseteq X$ which is dense and open in $\overline{X}$.*

In combination with Remark 1.7 this yields the result that the image of a morphism of affine varieties contains a subset that is dense and open in the image closure.

Remark. Algorithm 2.4 really works in all Noetherian topological spaces where intersections and unions of closed sets and closures of sets $A \setminus B$ with A, B closed can be computed explicitly. Thus Theorem 2.6 holds with K^n substituted by any Noetherian topological space. ◁

3 Affine schemes over fields

In this section we drop the assumption that K be algebraically closed. We write $\overline{K}$ for its algebraic closure. If the ideals and polynomials that form the input of the algorithms from Sections 1 and 2 are defined over K, then the computations will only involve coefficients from K. The results of the computations are constructible subsets of some affine n-space $\overline{K}^n$ over the algebraic closure. In this section we show that the results can also be interpreted in the scheme-theoretic sense over K.

Let $I_1, \ldots, I_r, A_1, \ldots, A_r \subseteq K[x_1, \ldots, x_n] =: K[\underline{x}]$ be ideals. Consider the constructible set

$$X := \bigcup_{i=1}^{r} \left(\mathrm{Var}_{\overline{K}^n}(I_i) \setminus \mathrm{Var}_{\overline{K}^n}(A_i)\right)$$

and its *scheme-theoretic counterpart*

$$X_s := \bigcup_{i=1}^{r} \left\{ P \in \mathrm{Spec}\,(K[\underline{x}]) \mid I_i \subseteq P \text{ and } A_i \not\subseteq P \right\}.$$

The following lemma connects X and X_s.

Lemma 3.1. *In the above situation, the following holds.*

(a) $X_s = \left\{ P \in \mathrm{Spec}\,(K[\underline{x}]) \mid \text{ there exists } \mathcal{M} \subseteq X \text{ with } P = \mathrm{Id}_{K[\underline{x}]}(\mathcal{M}) \right\}$.

(b) For every subset $\mathcal{M} \subseteq X$ there exist $P_1, \ldots, P_l \in X_s$ such that

$$\mathrm{Id}_{K[\underline{x}]}\,(\mathcal{M}) = \bigcap_{i=1}^{l} P_i.$$

(c) $\mathrm{Id}_{K[\underline{x}]}(X) = \bigcap_{P \in X_s} P.$

Proof. (a) Let $P \in X_s$. Then there exist $i \in \{1, \ldots, r\}$ and $h \in A_i$ such that $I_i \subseteq P$ and $h \notin P$. Set $\mathcal{M} := \mathrm{Var}_{\overline{K}^n}(P) \setminus \mathrm{Var}_{\overline{K}^n}(h)$. Then $\mathcal{M} \in X$ and $P \subseteq \mathrm{Id}_{K[\underline{x}]}(\mathcal{M})$. Conversely, take $g \in \mathrm{Id}_{K[\underline{x}]}(\mathcal{M})$. Then $hg \in \mathrm{Id}_{K[\underline{x}]}\,(\mathrm{Var}_{\overline{K}^n}(P))$. We have

$$\mathrm{Id}_{K[\underline{x}]}\,(\mathrm{Var}_{\overline{K}^n}(P)) = K[\underline{x}] \cap \mathrm{Id}_{\overline{K}[\underline{x}]}\left(\mathrm{Var}_{\overline{K}^n}\left(\langle P \rangle_{\overline{K}[\underline{x}]}\right)\right) = \sqrt{K[\underline{x}] \cap \langle P \rangle_{\overline{K}[\underline{x}]}} = \sqrt{P} = P,$$

where the second last equality follows since we have a $K[\underline{x}]$-linear projection $\overline{K}[\underline{x}] \to K[\underline{x}]$. We conclude $hg \in P$, so $g \in P$. Therefore $P = \mathrm{Id}_{K[\underline{x}]}(\mathcal{M})$, which proves one inclusion from (a). The reverse inclusion follows from (b)

(b) Let $A_i = \langle h_{i,1}, \ldots, h_{i,m_i} \rangle_{K[\underline{x}]}$. Then

$$\mathrm{Var}_{\overline{K}^n}(I_i) \setminus \mathrm{Var}_{\overline{K}^n}(A_i) = \bigcup_{i=1}^{m_i} \left(\mathrm{Var}_{\overline{K}^n}(I_i) \setminus \mathrm{Var}_{\overline{K}^n}(h_{i,j})\right).$$

Setting $\mathcal{M}_{i,j} := \mathcal{M} \cap (\mathrm{Var}_{\overline{K}^n}(I_i) \setminus \mathrm{Var}_{\overline{K}^n}(h_{i,j}))$, we obtain

$$\mathcal{M} = \bigcup_{i=1}^{r} \bigcup_{j=1}^{m_i} \mathcal{M}_{i,j} \quad \text{and} \quad \mathrm{Id}_{K[\underline{x}]}(\mathcal{M}) = \bigcap_{i=1}^{r} \bigcap_{j=1}^{m_i} \mathrm{Id}_{K[\underline{x}]}\,(\mathcal{M}_{i,j}).$$

So we may assume

$$\mathcal{M} \subseteq \mathrm{Var}_{\overline{K}^n}(I_i) \setminus \mathrm{Var}_{\overline{K}^n}(h) \quad \text{with} \quad i \in \{1, \ldots, r\} \quad \text{and} \quad h \in A_i. \tag{3.1}$$

Since $J := \mathrm{Id}_{K[\underline{x}]}(\mathcal{M})$ is a radical ideal, there exist $P_1, \ldots, P_l \in \mathrm{Spec}\,(K[\underline{x}])$ such that $J = \bigcap_{j=1}^{l} P_j$. By (3.1), we have $I_i \subseteq P_j$ for all j. Assume that $h \in P_1$, and let $g \in \bigcap_{j=2}^{l} P_j$. Then $hg \in J$. But h vanishes nowhere on $\mathcal{M}$, so $g \in J$. It follows that $J = \bigcap_{j=2}^{l} P_j$. This shows that we may assume that $h \notin P_j$ for all j. This implies $P_j \in X_s$.

(c) The inclusion $\mathrm{Id}_{K[\underline{x}]}(X) \subseteq \bigcap_{P \in X_s} P$ follows from (a), and the reverse inclusion follows from (b). $\square$

In the situation introduced before the lemma, let $f_1, \ldots, f_m \in K[\underline{x}]$, and consider the morphism

$$f\colon \overline{K}^n \to \overline{K}^m,\ (\xi_1, \ldots, \xi_n) \mapsto \left(f_1(\underline{\xi}), \ldots, f_m(\underline{\xi})\right)$$

and its scheme-theoretic counterpart

$$f_s\colon \operatorname{Spec}\left(K[x_1, \ldots, x_n]\right) \to \operatorname{Spec}\left(K[y_1, \ldots, y_m]\right)$$
$$P \mapsto \{F \in K[y_1, \ldots, y_m] \mid F(f_1, \ldots, f_m) \in P\}.$$

Assume that we have ideals $J_1, \ldots, J_l, B_1, \ldots, B_l \subseteq K[y_1, \ldots, y_m]$ such that

$$f(X) = \bigcup_{i=1}^{l} \left(\mathrm{Var}_{\overline{K}^m}(J_i) \setminus \mathrm{Var}_{\overline{K}^m}(B_i)\right).$$

The J_i and B_i may be the result of running Algorithm 1.6. The following theorem says that these ideals also describe the image in the scheme-theoretic sense. Write

$$Y_s := \bigcup_{i=1}^{l} \{Q \in \operatorname{Spec}\left(K[y_1, \ldots, y_m]\right) \mid J_i \subseteq Q \text{ and } B_i \not\subseteq Q\}$$

for the scheme-theoretic counterpart of $f(X)$.

Theorem 3.2. *In the above situation we have*

$$f_s(X_s) = Y_s.$$

Proof. To prove the first inclusion, take $P \in X_s$ and set $Q := f_s(P)$. By Lemma 3.1(a), there exists $\mathcal{M} \subseteq X$ with $P = \mathrm{Id}_{K[x_1,\ldots,x_n]}(\mathcal{M})$. Set $\mathcal{N} := f(\mathcal{M})$. Then

$$\mathrm{Id}_{K[y_1,\ldots,y_m]}(\mathcal{N}) = \left\{F \in K[y_1, \ldots, y_m] \mid F\left(f_1(\underline{\xi}), \ldots, f_m(\underline{\xi})\right) = 0 \text{ for all } (\underline{\xi}) \in \mathcal{M}\right\} =$$
$$\left\{F \in K[y_1, \ldots, y_m] \mid F(f_1, \ldots, f_m) \in \mathrm{Id}_{K[x_1,\ldots,x_n]}(\mathcal{M})\right\} = f_s(P) = Q.$$

Since $\mathcal{N} \subseteq f(X)$, it follows by Lemma 3.1(a) that $Q \in Y_s$.

Conversely, let $Q \in Y_s$. By Lemma 3.1(a), there exists $\mathcal{N} \subseteq f(X)$ with $Q = \mathrm{Id}_{K[y_1,\ldots,y_m]}(\mathcal{N})$, so $Q = \mathrm{Id}_{K[y_1,\ldots,y_m]}\left(f(\mathcal{M})\right)$ with $\mathcal{M} \subseteq X$. By Lemma 3.1(b), we have $P_1, \ldots, P_r \in X_s$ with $\mathrm{Id}_{K[x_1,\ldots,x_n]}(\mathcal{M}) = \bigcap_{i=1}^r P_i$. So

$$\bigcap_{i=1}^{r} f_s(P_i) = \left\{F \in K[y_1, \ldots, y_m] \mid F(f_1, \ldots, f_m) \in \mathrm{Id}_{K[x_1,\ldots,x_n]}(\mathcal{M})\right\} =$$
$$\mathrm{Id}_{K[y_1,\ldots,y_m]}\left(f(\mathcal{M})\right) = Q.$$

Therefore there exists i with $Q = f_s(P_i)$, so $Q \in f_s(X_s)$. $\square$

Let X and Y be two constructible subsets of $\overline{K}^n$, and let X_s and Y_s be their scheme-theoretic counterparts. From Lemma 3.1(a) we obtain

$$X \subseteq Y \quad \Rightarrow \quad X_s \subseteq Y_s,$$

and from Lemma 3.1(c) we obtain

$$\overline{X} \subseteq \overline{Y} \quad \Rightarrow \quad \overline{X_s} \subseteq \overline{Y_s}.$$

(The converse statements also hold, but we do not need them here.) From these implications it follows immediately that the results from running Algorithm 2.4 also carry over to the scheme-theoretic situation. This is the contents of the following theorem.

Theorem 3.3. *Let $X \subseteq \overline{K}^n$ be a constructible set with scheme-theoretic counterpart $X_s \subseteq \operatorname{Spec}\left(K[x_1, \ldots, x_n]\right)$. Moreover, let $U \subseteq X$ be a subset which is dense and open in $\overline{X}$. Then the scheme-theoretic counterpart U_s is contained in X_s, and U_s is dense and open in $\overline{X_s}$.*

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