KRULL DIMENSION AND MONOMIAL ORDERS

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Abstract. We introduce the notion of independent sequences with respect to a monomial order by using the least terms of polynomials vanishing at the sequence. Our main result shows that the Krull dimension of a Noetherian ring is equal to the supremum of the length of independent sequences. The proof has led to other notions of independent sequences, which have interesting applications. For example, we can show that $\dim R/0 : J^\infty$ is the maximum number of analytically independent elements in an arbitrary ideal J of a local ring R and that $\dim B \le \dim A$ if $B \subset A$ are (not necessarily finitely generated) subalgebras of a finitely generated algebra over a Noetherian Jacobson ring.

Introduction

Let R be an arbitrary Noetherian ring, where a ring is always assumed to be commutative with identity. The aim of this paper is to characterize the Krull dimension $\dim R$ by means of a monomial order on polynomial rings over R. We are inspired of a result of Lombardi in [13] (see also Coquand and Lombardi [4], [5]) which says that for a positive integer s, $\dim R < s$ if and only if for every sequence of elements $a_1, \dots, a_s$ in R, there exist nonnegative integers $m_1, \dots, m_s$ and elements $c_1, \dots, c_s \in R$ such that

$$a_1^{m_1} \cdots a_s^{m_s} + c_1 a_1^{m_1+1} + c_2 a_1^{m_1} a_2^{m_2+1} + \cdots + c_s a_1^{m_1} \cdots a_{s-1}^{m_{s-1}} a_s^{m_s+1} = 0.$$

This result has helped to develop a constructive theory for the Krull dimension [6], [7], [8].

The above relation means that $a_1, \dots, a_s$ is a solution of the polynomial

$$x_1^{m_1} \cdots x_s^{m_s} + c_1 x_1^{m_1+1} + c_2 x_1^{m_1} x_2^{m_2+1} + \cdots + c_s x_1^{m_1} \cdots x_{s-1}^{m_{s-1}} x_s^{m_s+1}.$$

The least term of this polynomial with respect to the lexicographic order is the monomial $x_1^{m_1} \cdots x_s^{m_s}$, which has the coefficient 1. This interpretation leads us to introduce the following notion.

Let $\prec$ be a monomial order on the polynomial ring $R[x_1, x_2, \dots]$ with infinitely many variables. For every polynomial f we write $\operatorname{in}_\prec(f)$ for the least term of f with respect to $\prec$. Let $R[X] = R[x_1, \dots, x_s]$. We call $a_1, \dots, a_s \in R$ a *dependent sequence* with respect to $\prec$ if there exists $f \in R[X]$ vanishing at $a_1, \dots, a_s$ such that the coefficient of $\operatorname{in}_\prec(f)$ is invertible. Otherwise, $a_1, \dots, a_s$ is called an *independent sequence* with respect to $\prec$.

Using this notion, we can reformulate Lombardi's result as $\dim R < s$ if and only if every sequence of elements $a_1, \dots, a_s$ in R is dependent with respect to the lexicographic order. Out of this reformulation arises the question whether one can replace the lexicographical monomial order by other monomial orders. The proof of Lombardi does not reveal how one can relate an arbitrary monomial order to the Krull dimension of the ring. We will give a positive answer to this question by proving that $\dim R$ is the supremum of the length of

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independent sequences for an arbitrary monomial order. This follows from Theorem 2.7 of this paper, which in fact strengthens the above statement. As an immediate consequence, we obtain other algebraic identities between elements of R than in Lombardi's result. Although our results are not essentially computational, the independence conditions can often be treated by computer calculations. For instance, using a short program written in MAGMA [2], the first author tested millions of examples which led to the conjecture that the above question has a positive answer [12]. The proof combines techniques of Gröbner basis theory and the theory of associated graded rings of filtrations. It has led to other notions of independent sequences which are of independent interest, as we shall see below.

Our idea is to replace the monomial order $\prec$ by a weighted degree on the monomials. Given an infinite sequence $\mathbf{w}$ of positive integers $w_1, w_2, \ldots$, we may consider $R[x_1, x_2, \ldots]$ as a weighted graded ring with $\deg x_i = w_i$, $i = 1, 2, \ldots$. For every polynomial f, we write $\text{in}_{\mathbf{w}}(f)$ for the weighted homogeneous part of f of least degree. We call $a_1, \ldots, a_s \in R$ a *weighted independent sequence* with respect to $\mathbf{w}$ if every coefficient of $\text{in}_{\mathbf{w}}(f)$ is not invertible for all polynomials $f \in R[X]$ vanishing at $a_1, \ldots, a_s$. Otherwise, $a_1, \ldots, a_s$ is called a *weighted dependent sequence* with respect to $\mathbf{w}$. We will see that if R is a local ring and $w_i = 1$ for all i, the sequence $a_1, \ldots, a_s$ is weighted independent if and only if the elements $a_1, \ldots, a_s$ are analytically independent, a basic notion in the theory of local rings. That is the reason why we use the terminology independent sequence for the above notions.

Let $Q = (x_1 - a_1, \ldots, x_s - a_s)$ be the ideal of polynomials of $R[X]$ vanishing at $a_1, \ldots, a_s$. Let $\text{in}_\prec(Q)$ and $\text{in}_{\mathbf{w}}(Q)$ denote the ideals of $R[X]$ generated by the polynomials $\text{in}_\prec(f)$ and $\text{in}_{\mathbf{w}}(f)$, $f \in Q$. We want to find a weight sequence $\mathbf{w}$ such that $\text{in}_\prec(Q) = \text{in}_{\mathbf{w}}(Q)$. It is well known in Gröbner basis theory that this can be done if $\text{in}_\prec(Q)$ and $\text{in}_{\mathbf{w}}(Q)$ were the largest term or the part of largest degree of f. In our setting we can solve this problem only if $\prec$ is Noetherian, that is, if every monomial has only a finite number of smaller monomials. In this case, $a_1, ..., a_s$ is independent with respect to $\mathbf{w}$ if and only if it is independent with respect to $\prec$. If $\prec$ is not Noetherian, we can still find a Noetherian monomial order $\prec'$ such that if $a_1, ..., a_s$ is independent with respect to $\prec$, then $a_1 a_i, ..., a_s a_i$ is independent with respect to $\prec'$ for some index i. By this way, we can reduce our investigation on the length of independent sequences to the weighted graded case.

We shall see that for every weight sequence $\mathbf{w}$, $\text{in}_{\mathbf{w}}(Q)$ is the defining ideal of the associated graded ring of certain filtration of R. Using properties of this associated graded ring we can show that the length of a weighted independent sequence is bounded above by $\dim R$, and that $a_1, \ldots, a_s$ is a weighted independent sequence if $\text{ht}(a_1, \ldots, a_s) = s$. From this it follows that $\dim R$ is the supremum of the length of independent sequences with respect to $\mathbf{w}$. This is formulated in more detail in Theorem 1.8 of this paper. Furthermore, we can also show that $\dim R / \cup_{n \geq 1} (0 : J^n)$ is the supremum of the length of weighted independent sequences in a given ideal J. If R is a local ring, this gives a characterization for the maximum number of analytically independent elements in J.

Since our results for independent sequences with respect to a monomial order and for weighted independent sequences are analogous, one may ask whether there is a common generalization. We shall see that there is a natural class of binary relations on the monomials which cover both monomial orders and weighted degrees and for which the modified statements of the above results still hold. We call such a relation a *monomial preorder*. The key point is to show that a monomial preorder $\prec$ can be approximated by a weighted degree sequence $\mathbf{w}$. This is somewhat tricky because $\mathbf{w}$ has to be chosen such that incomparable monomials with respect to $\prec$ have the same weighted degree. Since monomial preorders are not as strict as monomial orders, these results may find applications in computational problems.

For an algebra over a ring, we can extend the definition of independent sequences to give a generalization of the transcendence degree. Let A be an algebra over R. Given a monomial preorder $\prec$, we say that a sequence $a_1, \dots, a_s$ of elements of A is independent over R with respect to $\prec$ if for every polynomial $f \in R[X]$ vanishing at $a_1, \dots, a_n$, no coefficient of $\mathrm{in}_\prec(f)$ is invertible in R. If R is a field, this is just the usual notion of algebraic independence. In general, $\dim A$ is not the supremum of the length of independent sequences over R. However, if R is a Jacobson ring and A a subfinite R-algebra, that is, a subalgebra of a finitely generated R-algebra, we show that $\dim A$ is the supremum of the length of independent sequences with respect to $\prec$. So we obtain a generalization of the fundamental result that the transcendence degree of a finitely generated algebra over a field equals its Krull dimension. Our result has the interesting consequence that the Krull dimension cannot increase if one passes from a subfinite algebra over a Noetherian Jacobson ring to a subalgebra. For instance, if $H \subseteq G \subseteq \mathrm{Aut}(A)$ are groups of automorphisms of a finitely generated $\mathbb{Z}$-algebra A, then

$$\dim\left(A^G\right) \le \dim\left(A^H\right),$$

even though the invariant rings need not be finitely generated. We also show that the above properties characterize Jacobson rings.

The paper is organized as follows. In Sections 1 and 2 we investigate weighted independent sequences and independent sequences with respect to a monomial order. The extensions of these notions for monomial preorders and for algebras over a Jacobson ring will be treated in Sections 3 and 4, respectively.

We would like to mention that there exists an earlier version of this paper, titled "The Transcendence Degree over a Ring" and authored by the first author [12]. This earlier version will not be published since its results have merged into the present version.

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1. Weighted independent sequences

In this section we will prove some basic properties of weighted independent sequences and our aim is to show that the Krull dimension is the supremum of the length of weighted independent sequences.

Throughout this paper, let R be a Noetherian ring. Let $a_1, \dots, a_s$ be a sequence of nonzero elements of R, which are not invertible. Note that an element of R is weighted dependent if it is zero or invertible.

First, we shall see that weighted independent sequences are a generalization of analytically independent elements. Recall that if R is a local ring, the elements $a_1, \dots, a_s$ are called *analytically independent* if every homogeneous polynomial vanishing at $a_1, \dots, a_s$ has all its coefficients in the maximal ideal, which means that they are not invertible.

Let $\mathbf{w} = 1, 1, \dots$, the weight sequence with all $w_i = 1$. The weighted degree in this case is the usual degree. Hence $\mathrm{in}_{\mathbf{w}}(f)$ is the homogeneous part of smallest degree of a polynomial f. Thus, $a_1,\dots,a_s$ is analytically dependent if there exists a homogeneous polynomial vanishing at $a_1,\dots,a_s$ which has an invertible coefficient.

Example 1.1. Let a,b be two arbitrary integers. Since the greatest common divisor of a^2 and b^2 divides the product ab, there exist $c,d \in \mathbb{Z}$ such that $ab = ca^2 + db^2$. This relation shows that a,b is a weighted dependent sequence with respect to $\mathbf{w} = 1,1,\dots$

Set $R[X] = R[x_1,\dots,x_s]$. Let $f \in R[X]$ be an arbitrary polynomial vanishing at $a_1,\dots,a_s$ and $g = \operatorname{in}_{\mathbf{w}}(f)$, where $\mathbf{w} = 1,1,\dots$. Write every term u of f with $\deg u > \deg g$ in the form $u = hv$, where v is a monomial with $\deg v = \deg g$, and replace u by the term $h(a_1,\dots,a_s)v$. Then we obtain a homogeneous polynomial of the form $g+a_1g_1+\cdots+a_sg_s$ vanishing at $a_1,\dots,a_s$. If R is a local ring, the coefficients of g are not invertible if and only if the coefficients of $g + a_1g_1 + \cdots + a_sg_s$ are not invertible. Hence $a_1,\dots,a_s$ is a weighted independent sequence if and only if $a_1,\dots,a_s$ are analytically independent.

Unlike analytically independent elements, the notion of weighted independent sequences depends on the order of the elements if the weight sequence $\mathbf{w}$ contains some distinct numbers.

Example 1.2. Let $R = K[u,v]$ be a polynomial ring in two indeterminates over a ring K. The sequence uv, v is dependent with respect to the weights $1,2$ because $x_1 - ux_2$ vanishes at uv, v and $\operatorname{in}_{\mathbf{w}}(x_1 - ux_2) = x_1$. On the other hand, the sequence v, uv is independent with respect to the same weights. To see this let $f = (x_1-v)g+(x_2-uv)h$ be an arbitrary polynomial of $R[x_1,x_2]$ vanishing at v, uv. If $vg + uvh \neq 0$, $\operatorname{in}_{\mathbf{w}}(f) = -\operatorname{in}_{\mathbf{w}}(vg + uvh)$, whose coefficients are divided by v, hence not invertible. If $vg + uvh = 0$, $g = uh$ and $\operatorname{in}_{\mathbf{w}}(f) = \operatorname{in}_{\mathbf{w}}(x_1uh + x_2h) = \operatorname{in}_{\mathbf{w}}(x_1uh)$ since $\deg x_1 = 1 < 2 = \deg x_2$. All coefficients of $\operatorname{in}_{\mathbf{w}}(x_1uh)$ are divided by u, hence not invertible.

Let $\mathbf{w}$ be an arbitrary weight sequence. Let $Q = (x_1 - a_1,\dots,x_s - a_s)$, the ideal of polynomials of $R[X]$ vanishing at $a_1,\dots,a_s$. Let C be the set of the coefficients of all polynomials $\operatorname{in}_{\mathbf{w}}(f)$, $f \in Q$. It is easy to see that C is an ideal. Therefore, $a_1,\dots,a_s$ is a weighted independent sequence with respect to $\mathbf{w}$ if and only if C is a proper ideal of R. Using this characterization, we obtain the following property of weighted independent sequences under localization.

Proposition 1.3. *The sequence $a_1,\dots,a_s$ is weighted independent if and only if there is a prime P of R such that $a_1,\dots,a_s$ is weighted independent in R_P.*

Proof. If $a_1,\dots,a_s$ is a weighted independent sequence, then C is contained in a maximal ideal P of R. Since Q_P is the ideal of the polynomials in $R_P[X]$ vanishing at $a_1,\dots,a_s$, C_P is the set of the coefficients of all polynomials $\operatorname{in}_{\mathbf{w}}(f)$, $f \in Q_P$. Since C_P is a proper ideal of R_P, $a_1,\dots,a_s$ is a weighted independent sequence in R_P.

Conversely, if $a_1,\dots,a_s$ is a weighted independent sequence in R_P for some prime P of R, then C_P is a proper ideal and so is C, too. Therefore, $a_1,\dots,a_s$ is a weighted independent sequence in R. $\square$

Let $\operatorname{in}_{\mathbf{w}}(Q)$ denote the ideal in $R[X]$ generated by the polynomials $\operatorname{in}_{\mathbf{w}}(f)$, $f \in Q$. Then C is also the set of the coefficients of all polynomials in $\operatorname{in}_{\mathbf{w}}(Q)$. Therefore, weighted independence is a property of $\operatorname{in}_{\mathbf{w}}(Q)$. We shall see that $R[X]/\operatorname{in}_{\mathbf{w}}(Q)$ is isomorphic to the associated graded ring of certain filtration of R.

Let S denote the subring $R[a_1t^{w_1}, \dots, a_st^{w_s}, t^{-1}]$ of the Laurent polynomial ring $R[t, t^{-1}]$. Since S is a graded subring of $R[t, t^{-1}]$, we may write $S = \oplus_{n \in \mathbb{Z}} I_n t^n$. It is easy to see that

$$I_n = \sum_{\substack{m_1w_1 + \cdots + m_sw_s \geq n \\ m_1, \dots, m_s \geq 0}} a_1^{m_1} \cdots a_s^{m_s} R \tag{1.1}$$

for $n \geq 0$ and $I_n = R$ for $n < 0$. The ideals I_n, $n \geq 0$, form a filtration of R. In the case $w_1 = \cdots = w_s = 1$, we have $I_n = I^n$, where $I := (a_1, \dots, a_s)$. So we may consider S as the extended Rees algebra of this filtration.

Let $G = S/t^{-1}S$. Then $G \cong \oplus_{n \geq 0} I_n/I_{n+1}$. In other words, G is the associated graded ring of the above filtration.

Lemma 1.4. *$G \cong R[X]/\operatorname{in}_{\mathbf{w}}(Q)$.*

Proof. Let y be a new variable and consider the polynomial ring $R[X, y]$ as weighted graded with $\deg x_i = w_i$ and $\deg y = -1$. Then we have a natural graded map $R[X, y] \to S$, which sends x_i to $a_it^{w_i}$, $i = 1, \dots, s$, and y to t^{-1}. Let $\Im$ denote the kernel of this map. Then $S \cong R[X, y]/\Im$, hence

$$G \cong R[X, y]/(\Im, y) \cong R[X]/\left((\Im, y) \cap R[X]\right).$$

It remains to show that $(\Im, y) \cap R[X] = \operatorname{in}_{\mathbf{w}}(Q)$.

Let g be an arbitrary element of $(\Im, y) \cap R[X]$. Then $g = F + Hy$ for some polynomials $F \in \Im$ and $H \in R[X, y]$. Without loss of generality we may assume that g is nonzero and that g and F are weighted homogeneous. Then F has the form $F = g + g_1y + \cdots + g_ny^n$, where g_i is a weighted homogeneous polynomial of $R[X]$ with $\deg g_i = \deg g + i$, $i = 1, \dots, n$. Set $f = g + g_1 + \cdots + g_n$. We have $f(a_1, \dots, a_s)t^{\deg g} = F(a_1t^{w_1}, \dots, a_st^{w_s}, t^{-1}) = 0$. Therefore, $f(a_1, \dots, a_s) = 0$ and hence $f \in Q$. Since $g = \operatorname{in}_{\mathbf{w}}(f)$, $g \in \operatorname{in}_{\mathbf{w}}(Q)$.

Conversely, every polynomial $f \in Q$ can be written in the form $f = g + g_1 + \cdots + g_n$, where $g = \operatorname{in}_{\mathbf{w}}(f)$ and g_i is a weighted homogeneous polynomial with $\deg g_i = \deg g + i$, $i = 1, \dots, n$. Set $F = g + g_1y + \cdots + g_ny^n$. Then $F(a_1t^{w_1}, \dots, a_st^{w_s}, t^{-1}) = f(a_1, \dots, a_s)t^{\deg g} = 0$. Therefore $F \in \Im$ and hence $\operatorname{in}_{\mathbf{w}}(f) = F - (g_1 + \cdots + g_ny^{n-1})y \in (\Im, y)$. ☐

Corollary 1.5. *If $a_1, ..., a_s$ is a weighted independent sequence, then $s \leq \dim G$.*

Proof. Since $\operatorname{in}_{\mathbf{w}}(Q) \subseteq CR[X]$, there is a surjective map $R[X]/\operatorname{in}_{\mathbf{w}}(Q) \to R[X]/CR[X]$. Since C is a proper ideal of R, $s \leq \dim R[X]/CR[X]$ because $R[X]/CR[X] \cong (R/C)[X]$, the polynomial ring in s variables over R/C. Thus, $s \leq \dim R[X]/\operatorname{in}_{\mathbf{w}}(Q) = \dim G$. ☐

The following formula for $\dim G$ follows from a general formula for the dimension of the associated graded ring of a filtration [3, Theorem 4.5.6(b)]. This formula is a generalization of the well-known fact that $\dim G = \dim R$ if R is a local ring and G is the associated graded ring of an ideal (see Matsumura [14, Theorem 15.7] or Eisenbud [9, Excercise 13.8]).

Lemma 1.6. *Let $I = (a_1, \dots, a_s)$. Then*

$$\dim G = \sup\{\operatorname{ht} P \mid P \supseteq I \text{ is a prime of } R\}.$$

As a consequence, we always have $\dim G \leq \dim R$. Together with Corollary 1.5, this implies that the length of a weighted independent sequence cannot exceed $\dim R$. Now we will show that there exist weighted independent sequences of length $\operatorname{ht} P$ for any maximal prime P of R.

Let $\operatorname{bight}(I)$ denote the *big height* of I, that is, the maximum height of the minimal primes over I.

Proposition 1.7. *Let $a_1, \dots, a_s$ be elements of R such that* $\operatorname{bight}(a_1, \dots, a_s) = s$. *Then $a_1, \dots, a_s$ is a weighted independent sequence with respect to every weight sequence* $\mathbf{w}$.

Proof. Let P be a minimal prime of $I = (a_1, \dots, a_s)$ with $\operatorname{ht} P = s$. By Proposition 1.3, $a_1, \dots, a_s$ is a weighted independent sequence in R if $a_1, \dots, a_s$ is a weighted independent sequence in R_P. Therefore, we may assume that R is a local ring and $a_1, \dots, a_s$ is a system of parameters in R. In this case, $\dim G = s$ by Lemma 1.6.

Let $\mathfrak{m}$ be the maximal ideal of R. There exists an integer r such that $\mathfrak{m}^r \subseteq I$. Since $I_1 = I$, $\mathfrak{m}^r I_n \subseteq I_{n+1}$ for all n. Therefore, $\mathfrak{m}^r G = 0$. Hence $\dim G/\mathfrak{m}G = \dim G = s$. Let $k = R/\mathfrak{m}$. By Lemma 1.4, $G/\mathfrak{m}G = R[X]/(\operatorname{in}_{\mathbf{w}}(Q), \mathfrak{m}) = k[X]/J$ for some ideal J of $k[X]$. If $a_1, \dots, a_s$ were weighted dependent, there would be a polynomial in $\operatorname{in}_{\mathbf{w}}(Q)$ which has a coefficient not in $\mathfrak{m}$, implying $J \neq 0$ and the contradiction $\dim(G/\mathfrak{m}G) \leq s - 1$. □

Summing up, we obtain the following results on the Krull dimension in terms of weighted independent sequences.

Theorem 1.8. *Let R be a Noetherian ring and s a positive integer.*

(a) *If $s \leq \dim R$, there exists a sequence $a_1, \dots, a_s \in R$ that is weighted independent with respect to every weight sequence.*
(b) *If $s > \dim R$, every sequence $a_1, \dots, a_s \in R$ is weighted dependent with respect to every weight sequence.*

Proof. If $s \leq \dim R$, there exists a prime P in R of height s. It is a standard fact that there exists a sequence $a_1, \dots, a_s \in P$ such that P is a minimal prime of $(a_1, \dots, a_s)$. Hence (a) follows from Proposition 1.7. If $s > \dim R$, then $s > \dim G$ by Lemma 1.6. Hence (b) follows from Corollary 1.5. □

As a consequence, $\dim R$ is the supremum of the length of weighted independent sequences with respect to an arbitrary weight sequence.

Remark. A maximal weighted independent sequence need not to have length $\dim R$. To see that we consider a Noetherian ring that has a maximal ideal $P = (a_1, ..., a_s)$ with $s = \operatorname{ht} P < \dim R$. By Proposition 1.7, $a_1, \dots, a_s$ is weighted independent with respect to every weight sequence $\mathbf{w}$. It is maximal because any extended sequence $a_1, ..., a_{s+1}$ with $a_{s+1} \notin P$ is weighted dependent. This follows from the fact $R = (a_1, ..., a_{s+1})$, which implies that there is a polynomial f of the form $1 + c_1 x_1 + \cdots + c_{s+1} x_{s+1}$ vanishing at $a_1, ..., a_{s+1}$ with $\operatorname{in}_{\mathbf{w}}(f) = 1$.

Similarly, we can study weighted independent sequences in a given ideal J of R. Let $0 : J^\infty = \cup_{m \geq 0} 0 : J^m$. Note that $0 : J^\infty$ is the intersection of all primary components of the zero-ideal 0_R whose associated primes do not contain J and that $0 : J^\infty = 0 : J^m$ for m large enough.

Theorem 1.9. *For every ideal $J \subseteq R$, $\dim R/0 : J^\infty$ is the supremum of the length of weighted independent sequences in J with respect to an arbitrary weight sequence.*

Proof. Let P be a maximal prime of $R/0 : J^\infty$ and $s = \operatorname{ht} P$. Using Proposition 1.7 we can find elements $a_1, \dots, a_d$ in R such that their residue classes in $R/0 : J^\infty$ is a weighted independent sequence. Choose $c \in J$ such that c is not contained in any associated prime of 0_R not containing J. Then $0 : c^\infty = 0 : J^\infty$. We claim that $a_1 c^{w_1}, \dots, a_s c^{w_s}$ is a weighted independent sequence. To see this let f be a polynomial in $R[X]$ vanishing at

$a_1c^{w_1},\dots,a_sc^{w_s}$ and $r = \deg \mathrm{in}_{\mathbf{w}}(f)$. Write f in the form $f = \mathrm{in}_{\mathbf{w}}(f)+g_1+\cdots+g_n$, where g_i is a weighted homogeneous polynomial of degree $r+i$, $i=1,\dots,n$. Then

$$\begin{aligned}&f(a_1c^{w_1},\dots,a_sc^{w_s})\\&= c^r\,\mathrm{in}_{\mathbf{w}}(f)(a_1,\dots,a_s)+c^{r+1}g_1(a_1,\dots,a_s)+\cdots+c^{r+n}g_n(a_1,\dots,a_s)=0.\end{aligned}$$

Therefore, if we put $h = \mathrm{in}_{\mathbf{w}}(f)+cg_1+\cdots+c^ng_n$, then $h(a_1,\dots,a_s)\in 0:c^d\subseteq 0:J^\infty$ and $\mathrm{in}_{\mathbf{w}}(h)=\mathrm{in}_{\mathbf{w}}(f)$. By the choice of $a_1,\dots,a_s$, the coefficients of $\mathrm{in}_{\mathbf{w}}(f)$ cannot not be invertible. This shows the existence of a weighted independent sequence of length s in J. Hence $\dim R/0:J^\infty$ is less than or equal to the supremum of the length of weighted independent sequences in J.

Now we will show that $s\le \dim R/0:J^\infty$ for any weighted independent sequence $a_1,\dots,a_s$ in J. Let m be a positive number such that $0:J^\infty = 0:J^m$. Then $(0:J^\infty)a_i^m=0$, $i=1,\dots,s$. This implies $(0:J^\infty)x_i^m\subseteq \mathrm{in}_{\mathbf{w}}(Q)$. Hence $0:J^\infty\subseteq C$, where C is the ideal of the coefficients of polynomials in $\mathrm{in}_{\mathbf{w}}(Q)$. Let $\mathfrak{m}$ be a maximal ideal of R containing C. Then $\mathrm{in}_{\mathbf{w}}(Q)+(0:J^\infty)R[X]\subseteq \mathfrak{m}R[X]$. By Lemma 1.6 there is a surjective map $G/(0:J^\infty)G\to R[X]/\mathfrak{m}R[X]$. From this it follows that $s=\dim R[X]/\mathfrak{m}R[X]\le \dim G/(0:J^\infty)G$. We have

$$G/(0:J^\infty)G = \oplus_{n\ge0}I_n/\big((0:J^\infty)I_n+I_{n+1}\big),$$

and we will compare this with the ring

$$G' := \oplus_{n\ge0}\big(I_n+(0:I^\infty)\big)/\big(I_{n+1}+(0:I^\infty)\big),$$

which is the associated graded ring of $R/0:J^\infty$ with respect to the filtration $\big(I_n+(0:J^\infty)\big)/(0:J^\infty)$, $n\ge 0$. Note that $\dim G'\le \dim R/0:J^\infty$ by Lemma 1.6. Then $s\le \dim R/0:J^\infty$ if we can show that $\dim G/(0:J^\infty)G=\dim G'$.

Let $w_{\max}:=\max\{w_i|\ i=1,\dots,s\}$. By Equation (1.1) we have $I_n\subseteq I^m$ for $n\ge mw_{\max}$. This implies $(0:J^\infty)I_n\subseteq(0:J^\infty)I^m=0$. Using Artin-Rees lemma we can also show that $(0:J^\infty)\cap I_n=0$ for n large enough. Thus, there exists a positive number r such that $(0:J^\infty)I_n=(0:J^\infty)\cap I_n=0$ for $n\ge r$. This relation implies

$$I_n/\big((0:J^\infty)I_n+I_{n+1}\big)=I_n/\big((0:J^\infty)\cap I_n+I_{n+1}\big)\cong\big(I_n+(0:J^\infty)\big)/\big(I_{n+1}+(0:J^\infty)\big).$$

Hence

$$\bigoplus_{n\ge0}I_{nr}/\big((0:J^\infty)I_{nr}+I_{nr+1}\big)\cong\bigoplus_{n\ge0}\big(I_{nr}+(0:J^\infty)\big)/\big(I_{nr+1}+(0:J^\infty)\big).$$

The graded rings on both sides are Veronese subrings of $G/(0:J^\infty)G$ and G', respectively. Since the dimension of a Veronese subring is the same as of the original ring, we get $\dim G/(0:J^\infty)G=\dim G'$, as required. $\square$

Theorem 1.9 has the following immediate consequence.

Corollary 1.10. *Let R be a local ring and J an ideal of R. Then $\dim R/0:J^\infty$ is the maximum number of analytically independent elements in J.*

This result seems to be new though there was a general (but complicated) formula for the maximum number of $\mathfrak{a}$-independent elements in J, where $\mathfrak{a}$ is an ideal containing J (see [1], [17]). Recall that the elements $a_1,\dots,a_s$ are called $\mathfrak{a}$-independent if every homogeneous form in $R[X]$ vanishing at $a_1,\dots,a_s$ has all its coefficients in $\mathfrak{a}$. This notion was introduced by Valla [18].

2. Independent sequences with respect to a monomial order

In this section we will show how to approximate a monomial order by a weighted degree and we will prove that the Krull dimension is the supremum of the length of independent sequences with respect to an arbitrary monomial order.

Let $a_1,\dots,a_s$ be elements of a Noetherian ring R. Recall that $a_1,\dots,a_s$ is a *dependent sequence* with respect to a monomial order $\prec$ if there exists $f \in R[x_1,...,x_s]$ vanishing at $a_1,\dots,a_s$ such that the coefficient of $\mathrm{in}_\prec(f)$ is invertible. Otherwise, $a_1,\dots,a_s$ is called an *independent sequence* with respect to $\prec$.

The following example suggests that dependence with respect to a monomial order is more subtle than weighted dependence.

Example 2.1. Let $R=\mathbb{Z}$ and let $\prec$ be the lexicographic order with $x_1 \succ x_2$. Clearly the single elements that are dependent with respect to $\prec$ are 0 and the invertible elements. We claim that a sequence of two arbitrary integers a,b is always dependent with respect to $\prec$. The relation $ab = ca^2+db^2$ found in Example 1.1 does not show the dependence, so we have to argue in a different way. We may assume a and b to be nonzero and write

$$a = \pm\prod_{i=1}^r p_i^{d_i} \quad \text{and} \quad b = \pm\prod_{i=1}^r p_i^{e_i},$$

where the p_i are pairwise distinct prime numbers and $d_i, e_i \in \mathbb{N}_0$. Choose $n \in \mathbb{N}_0$ such that $n \geq d_i/e_i$ for all i with $e_i > 0$. Then

$$\gcd(a, b^{n+1}) = \prod_{i=1}^r p_i^{\min\{d_i,(n+1)e_i\}} \quad \text{divides} \quad \prod_{i=1}^r p_i^{ne_i} = b^n,$$

so there exist $c,d\in\mathbb{Z}$ such that $b^n = ca+db^{n+1}$. Since the least term of $f = x_2^n - cx_1 - dx_2^{n+1}$ is x_2^n this relation shows that a,b are dependent, as claimed.

The argument can easily be adapted to any Dedekind domain.

It is easy to see that the notion of independent sequence depends on the order of the elements. For instance, the sequence uv, v of Example 1.2 is independent with respect to the lexicographic order, while v, uv is not by using the same arguments.

Set $R[X] = R[x_1,\dots,x_s]$ and $Q = (x_1-a_1,\dots,x_s-a_s)$, the ideal of all polynomials of $R[X]$ vanishing at $a_1,\dots,a_s$. Let $\mathrm{in}_\prec(Q)$ denote the ideal generated by the terms $\mathrm{in}_\prec(f)$, $f\in Q$. One may ask whether there exists a weight sequence $\mathbf{w}$ such that $\mathrm{in}_\mathbf{w}(Q) = \mathrm{in}_\prec(Q)$. For this will imply that $a_1,\dots,a_s$ is an independent sequence with respect to $\prec$ if and only if it is a weighted independent sequence with respect to $\mathbf{w}$.

To study this problem we need the following result in Gröbner basis theory.

Lemma 2.2 (see Eisenbud [9, Exercise 15.12], [11, Exercise 9.2(b)])**.** *Let $\mathcal{M}$ be a finite set of polynomials. Then there exists a weight sequence $\mathbf{w}$ such that $\mathrm{in}_\prec(f) = \mathrm{in}_\mathbf{w}(f)$ for all $f \in \mathcal{M}$.*

We call $\prec$ a *Noetherian monomial order* if for every monomial $f \in R[X]$ there are only finitely many monomials $g \in R[X]$ with $g \prec f$. This class of monomial orders is rather large. For instance, every monomial order that first compares the (weighted) degree of the monomials is Noetherian.

Proposition 2.3. *For every ideal $\Im$ of $R[X]$, there exists a weight sequence $\mathbf{w}$ such that $\mathrm{in}_\prec(\Im) \subseteq \mathrm{in}_\mathbf{w}(\Im)$. If $\prec$ is Noetherian, $\mathbf{w}$ can be chosen such that $\mathrm{in}_\prec(\Im) = \mathrm{in}_\mathbf{w}(\Im)$.*

Proof. Choose $g_1, \dots, g_r \in \Im$ such that $\mathrm{in}_\prec(\Im) = \big(\mathrm{in}_\prec(g_1), \dots, \mathrm{in}_\prec(g_r)\big)$. From Lemma 2.2 it follows that there exists a weight sequence $\mathbf{w}$ such that $\mathrm{in}_\prec(g_i) = \mathrm{in}_\mathbf{w}(g_i)$ for all $i = 1, \dots, r$. This implies the first assertion:

$$\mathrm{in}_\prec(\Im) = \big(\mathrm{in}_\mathbf{w}(g_1), \dots, \mathrm{in}_\mathbf{w}(g_r)\big) \subseteq \mathrm{in}_\mathbf{w}(\Im).$$

Now we will assume that $\prec$ is Noetherian and prove equality. By way of contradiction, assume that there exists a polynomial $f \in \Im$ such that $\mathrm{in}_\mathbf{w}(f) \notin \mathrm{in}_\prec(\Im)$. Choose f such that $\mathrm{in}_\mathbf{w}(f)$ has the least possible number of terms. For every $g \in R[X]$ we have $\mathrm{in}_\prec(g) \preceq \mathrm{in}_\prec(\mathrm{in}_\mathbf{w}(g))$, so $\mathrm{in}_\prec(\mathrm{in}_\mathbf{w}(f))$ is an upper bound for all initial terms of polynomials g with $\mathrm{in}_\mathbf{w}(g) = \mathrm{in}_\mathbf{w}(f)$. By the assumption on the monomial order, we can therefore choose f such that for all $g \in \Im$,

$$\mathrm{in}_\mathbf{w}(g) = \mathrm{in}_\mathbf{w}(f) \quad \text{implies} \quad \mathrm{in}_\prec(g) \preceq \mathrm{in}_\prec(f). \tag{2.1}$$

Since $\mathrm{in}_\prec(f) \in \mathrm{in}_\prec(\Im)$, we have $\mathrm{in}_\prec(f) = h_1 \,\mathrm{in}_\prec(g_1) + \cdots + h_r \,\mathrm{in}_\prec(g_r)$ for some polynomials $h_1, \dots, h_r$. By deleting some terms of the h_i, we may assume that either $h_i = 0$ or h_i is a term such that $h_i \,\mathrm{in}_\prec(g_i)$ and $\mathrm{in}_\prec(f)$ are R-multiples of the same monomial. Set $h = h_1 g_1 + \cdots + h_r g_r \in \Im$. Then

$$\mathrm{in}_\prec(f) = \mathrm{in}_\prec(h) = \mathrm{in}_\mathbf{w}(h), \tag{2.2}$$

where the second equality follows from $\mathrm{in}_\prec(g_i) = \mathrm{in}_\mathbf{w}(g_i)$. For $g := f - h \in \Im$, this implies $\mathrm{in}_\prec(g) \succ \mathrm{in}_\prec(f)$, so $\mathrm{in}_\mathbf{w}(g) \neq \mathrm{in}_\mathbf{w}(f)$ by (2.1). We also have $\mathrm{in}_\mathbf{w}(h) \neq \mathrm{in}_\mathbf{w}(f)$ because otherwise $\mathrm{in}_\mathbf{w}(f) = \mathrm{in}_\prec(f) \in \mathrm{in}_\prec(I)$ by (2.2). For the weighted degrees we have the inequality

$$\deg\left(\mathrm{in}_\mathbf{w}(f)\right) \leq \deg\left(\mathrm{in}_\prec(f)\right) = \deg\left(\mathrm{in}_\mathbf{w}(h)\right).$$

In combination with $\mathrm{in}_\mathbf{w}(g) \neq \mathrm{in}_\mathbf{w}(f) \neq \mathrm{in}_\mathbf{w}(h)$, this implies that $\mathrm{in}_\mathbf{w}(f)$, $\mathrm{in}_\mathbf{w}(h)$, and $\mathrm{in}_\mathbf{w}(g)$ all have the same degree. So $\mathrm{in}_\mathbf{w}(g) = \mathrm{in}_\mathbf{w}(f) - \mathrm{in}_\mathbf{w}(h)$. By (2.2), subtracting $\mathrm{in}_\mathbf{w}(h)$ from $\mathrm{in}_\mathbf{w}(f)$ removes the initial term of $\mathrm{in}_\mathbf{w}(f)$ but leaves all other terms unchanged. So $\mathrm{in}_\mathbf{w}(g)$ has fewer terms than $\mathrm{in}_\mathbf{w}(f)$, and because of the choice of f we conclude $\mathrm{in}_\mathbf{w}(g) \in \mathrm{in}_\prec(\Im)$. But since $\mathrm{in}_\mathbf{w}(h) \in \mathrm{in}_\prec(\Im)$ by (2.2), this implies $\mathrm{in}_\mathbf{w}(f) \in \mathrm{in}_\prec(\Im)$, a contradiction. □

We do not know whether the Noetherian hypothesis is really necessary for the second assertion of Proposition 2.3.

Remark. For a polynomial $f \in R[X]$, we can also consider the leading term $\mathrm{LT}_\prec(f)$ and the weighted homogeneous part of highest degree, i.e., the *leading form* $\mathrm{LF}_\mathbf{w}(f)$. This defines $\mathrm{LT}_\prec(\Im)$ and $\mathrm{LF}_\mathbf{w}(\Im)$ for an ideal $\Im \subseteq R[X]$. Then Proposition 2.3 remains correct without the Noetherian hypothesis if we substitute $\mathrm{in}_\prec$ by $\mathrm{LT}_\prec$ and $\mathrm{in}_\mathbf{w}$ by $\mathrm{LF}_\mathbf{w}$. This is a well known result in Gröbner basis theory (see Eisenbud [9, Proposition 15.16] or [11, Exercise 9.2(c)]).

Proposition 2.3 implies the following relationship between weighted independent sequences and independent sequences with respect to monomial orders.

Corollary 2.4. *Let $a_1, \dots, a_s$ be a sequence of elements in R.*

(a) *If the sequence is weighted independent with respect to every weight sequence, it is independent with respect to every monomial order.*
(b) *If the sequence is weighted dependent with respect to every weight sequence, it is dependent with respect to every Noetherian monomial order.*

Proof. By Proposition 2.3 there exists a weight sequence $\mathbf{w}$ such that $\mathrm{in}_{\prec}(Q) \subseteq \mathrm{in}_{\mathbf{w}}(Q)$, with equality if $\prec$ is Noetherian. Under the hypothesis of (a) there exists a maximal ideal $\mathfrak{m} \subset R$ such that $\mathrm{in}_{\mathbf{w}}(Q) \subseteq \mathfrak{m}R[X]$, so $\mathrm{in}_{\prec}(Q) \subseteq \mathfrak{m}R[X]$. This shows that the sequence is independent with respect to $\prec$.

Under the hypothesis of (b), the ideal C of coefficients of polynomials in $\mathrm{in}_{\mathbf{w}}(Q)$ is R. Since $\mathrm{in}_{\mathbf{w}}(Q) = \mathrm{in}_{\prec}(Q)$, this implies that the sequence is dependent with respect to $\prec$. $\square$

We will get rid of the Noetherian hypothesis on a monomial order by showing that an independent sequence with respect to an arbitrary monomial order can be converted to an independent sequence of the same length with respect to a Noetherian monomial order. To do that we shall need Robbiano's characterization of monomial orders.

Lemma 2.5 ([15])**.** *For every monomial order $\prec$ in s variables, there exists a real matrix M having s rows such that $x_1^{m_1} \cdots x_s^{m_s} \prec x_1^{m'_1} \cdots x_s^{m'_s}$ if and only if*

$$(m_1, \dots, m_s) \cdot M <_{\mathrm{lex}} (m'_1, \dots, m'_s) \cdot M,$$

where $<_{\mathrm{lex}}$ is the lexicographic order. Moreover, the first column of M is nonzero and all its entries are nonnegative.

Proposition 2.6. *Let $a_1, \dots, a_s \in R$ be an independent sequence with respect to an arbitrary monomial order $\prec$. Then there exists an index i such that the sequence $a_1a_i, \dots, a_sa_i$ is independent with respect to some Noetherian monomial order $\prec'$.*

Proof. By Lemma 2.5, there exists a real vector $(v_1, \dots, v_s)$ having nonnegative components with some $v_i > 0$ such that $x_1^{m_1} \cdots x_s^{m_s} \prec x_1^{m'_1} \cdots x_s^{m'_s}$ implies $\sum_{j=1}^s m_j v_j \le \sum_{j=1}^s m'_j v_j$. Define $\prec'$ by the rule:

$$x_1^{m_1} \cdots x_s^{m_s} \prec' x_1^{m'_1} \cdots x_s^{m'_s} \quad \text{if} \quad (x_1x_i)^{m_1} \cdots (x_sx_i)^{m_s} \prec (x_1x_i)^{m'_1} \cdots (x_sx_i)^{m'_s}.$$

It is easy to see that $\prec'$ is a monomial order. If $x_1^{m_1} \cdots x_s^{m_s} \prec' x_1^{m'_1} \cdots x_s^{m'_s}$, then $\sum_{j=1}^s m_j(v_i + v_j) \le \sum_{i=1}^s m'_j(v_i + v_j)$. Since $v_i + v_j > 0$ for all $j = 1, \dots, s$, $\prec'$ is Noetherian.

Let f be a polynomial in $R[X]$ such that $f(a_1a_i, \dots, a_sa_i) = 0$. Put $g = f(x_1x_i, \dots, x_sx_i)$. Then $\mathrm{in}_{\prec}(g)$ has the same coefficient as $\mathrm{in}_{\prec'}(f)$. Since $g(a_1, \dots, a_s) = 0$, the coefficient of $\mathrm{in}_{\prec}(g)$ is not invertible. This shows that the coefficient of $\mathrm{in}_{\prec'}(f)$ is not invertible. $\square$

Now we are ready to extend Lombardi's characterization of the Krull dimension to an arbitrary monomial order.

Theorem 2.7. *Let R be a Noetherian ring and s a positive integer.*

(a) *If $s \le \dim R$, there exists a sequence $a_1, \dots, a_s \in R$ that is independent with respect to every monomial order.*
(b) *If $s > \dim R$, every sequence $a_1, \dots, a_s \in R$ is dependent with respect to every monomial order.*

Proof. If $s \le \dim R$, there exists a sequence $a_1, \dots, a_s \in R$ which is weighted independent with respect to every weight sequence by Theorem 1.8(a). By Corollary 2.4(a), this implies that $a_1, \dots, a_s$ is independent with respect to every monomial order.

If $s > \dim R$, every sequence $a_1, \dots, a_s \in R$ is weighted dependent with respect to every weight sequence by Theorem 1.8(b). If $a_1, \dots, a_s$ is independent for some monomial order, then $a_1a_i, \dots, a_sa_i$ is independent with respect to some Noetherian monomial order for some i by Proposition 2.6. By Corollary 2.4(b), $a_1a_i, \dots, a_sa_i$ is weighted independent with respect to some weight sequence, a contradiction. $\square$

As a consequence, $\dim R$ is the supremum of the length of independent sequences with respect to an arbitrary monomial order. In the following we show how this result can be used to prove the existence of certain relations which look like polynomial identities in R.

Let $\prec$ be an arbitrary monomial order. For every term g of $R[X]$ there is a unique set $\mathcal{M}(g)$ of monomials $h \succ g$ such that

(i) every monomial $u \succ g$ is divisible by a monomial of $\mathcal{M}(g)$,
(ii) the monomials of $\mathcal{M}(g)$ are not divisible by each other.

For every polynomial $f \in R[X]$ vanishing at $a_1, \dots, a_s$, we can always find a polynomial vanishing at $a_1, \dots, a_s$ of the form

$$g + \sum_{h \in \mathcal{M}(g)} c_h h$$

where $g = \mathrm{in}_\prec(f)$ and $c_h \in R$. To see this, one only needs to write every term $u \succ g$ of f in the form $u = vh$ for some $h \in \mathcal{M}(g)$ and replace u by the term $v(a_1, \dots, a_s)h$. Therefore, $a_1, \dots, a_s$ is a dependent sequence with respect to $\prec$ if and only if there exists a polynomial of the above form vanishing at $a_1, \dots, a_s$ such that the coefficient of g is 1. Since the monomials of $\mathcal{M}(g)$ can be written down in a canonical way from the exponent vector of g, this polynomial yields an algebraic relation between elements of R which are similar to a polynomial identity.

Example 2.8. Let $\prec$ be the lexicographic order. For a monomial $g = x_1^{m_1} \cdots x_s^{m_s}$, $\mathcal{M}(g)$ is the set of the monomials $x_1^{m_1+1}, x_1^{m_1} x_2^{m_2+1}, \dots, x_1^{m_1} \cdots x_{s-1}^{m_{s-1}} x_s^{m_s+1}$. Therefore, $a_1, \dots, a_s$ is a dependent sequence with respect to the lexicographic order if and only if there exists a relation of the form

$$a_1^{m_1} \cdots a_s^{m_s} + c_1 a_1^{m_1+1} + c_2 a_1^{m_1} a_2^{m_2+1} + \cdots + c_s a_1^{m_1} \cdots a_{s-1}^{m_{s-1}} a_s^{m_s+1} = 0,$$

where $c_1, \dots, c_s \in R$. This explains why Theorem 2.7 is a generalization of Lombardi's result in [13]. In that paper Lombardi calls $a_1, \dots, a_s$ a pseudo-regular sequence if

$$a_1^{m_1} \cdots a_s^{m_s} + c_1 a_1^{m_1+1} a_2^{m_2} \cdots a_s^{m_s} + \cdots + c_s a_1^{m_1} \cdots a_{s-1}^{m_{s-1}} a_s^{m_s+1} \neq 0$$

for all nonnegative integers $m_1, \dots, m_s$ and $c_1, \dots, c_s \in R$. By the above observation, $a_1, \dots, a_s$ is pseudo-regular if and only if it is independent with respect to the lexicographic order.

Similarly as for weighted independent sequences, one may ask whether $\dim R/0 : J^\infty$ is the supremum of the length of independent sequences in an ideal $J \subseteq R$ with respect to an arbitrary monomial order. Unlike the case of weighted independent sequences, we could not give a full answer to this question. This shows again that independence with respect to a monomial order is more subtle than weighted independence.

Proposition 2.9. *Let J be an arbitrary ideal of R. The length of independent sequences in J with respect to an arbitrary monomial order is bounded above by* $\dim R/0 : J^\infty$.

Proof. Let $a_1, \dots, a_s$ be an independent sequence in J with respect to an arbitrary monomial order $\prec$. By Lemma 2.6, $a_1 a_i, \dots, a_s a_i$ is an independent sequence with respect to some Noetherian monomial order for some i. By Corollary 2.4, $a_1 a_i, \dots, a_s a_i$ is weighted independent for some weight sequence. Since $a_1 a_i, \dots, a_s a_i \in J$, $s \leq \dim R/0 : J^\infty$ by Theorem 1.9. $\square$

3. Generalization to monomial preorders

In the previous sections we have considered weight sequences and monomial orders, and shown analogous results in both cases. So one may ask whether there is a common generalization of these results. We shall see that the following notion provides the platform for such a generalization.

Recall that a *strict weak order* is a binary relation $\prec$ on a set M such that for $f, g, h \in M$ with $f \prec g$ we have:

(i) $f \prec h$ or $h \prec g$, and
(ii) $g \not\prec f$ (i.e., $g \prec f$ does not hold).

This is equivalent to say that $\prec$ is a strict partial order in which the incomparability relation (given by $f \not\prec g$ and $g \not\prec f$) is an equivalence relation and the equivalence classes of incomparable elements are totally ordered.

We call a strict weak order $\prec$ on the set of monomials of the variables $x_1, x_2, \dots$ a *monomial preorder* if it satisfies the following conditions:

(iii) $1 \prec f$ for all monomials $f \neq 1$, and
(iv) for all monomials f, g, h the equivalence

$$f \prec g \iff fh \prec gh$$

holds.

Notice that the actual preorder $\precsim$ is given by $f \precsim g \;\Leftrightarrow g \not\prec f$, not by $f \prec g$. This slight inaccuracy in terminology follows common practice in Gröbner basis theory.

Obviously, every monomial order is a monomial preorder. A weight sequence $\mathbf{w} = w_1, w_2, \dots$ gives rise to a preorder $\prec_{\mathbf{w}}$ by comparing their weighted degree, i.e.

$$\prod_i x_i^{m_i} \prec_{\mathbf{w}} \prod_i x_i^{m_i'} \quad \text{if} \quad \sum_i m_i w_i < \sum_i m_i' w_i.$$

We call this the $\mathbf{w}$-weighted preorder. The following example shows that monomial preorders are much more general than monomial orders and weighted preorders.

Example 3.1. Let M be a real matrix of s rows such that the first column is nonzero with nonnegative entries and every row is nonzero with the first nonzero entry positive. Then M defines a monomial preorder in a polynomial ring of s variables by

$$f \prec \; g \quad \text{if} \quad \exp(f) \cdot M <_{\text{lex}} \exp(g) \cdot M,$$

where f, g are monomials, $\exp(f)$ and $\exp(g)$ denote the exponent vectors of f, g, and $<_{\text{lex}}$ is the lexicographic order. Note that the assumption on the rows of M is equivalent to (iii). Then every monomial order arises in such a way by Lemma 2.5. If M has only one column and if its entries are positive integers, then we get a weighted preorder.

Lemma 3.2. *Every monomial preorder $\prec$ can be refined to a monomial order $\prec^*$, i.e. $f \prec g$ implies $f \prec^* g$.*

Proof. We choose an arbitrary monomial ordering $\prec'$ and use it to break ties in the equivalence classes of incomparable elements. More precisely, we define $f \prec^* g$ if $f \prec g$ or if f, g is incomparable and $f \prec' g$. It is straightforward to check that $\prec^*$ is a monomial order and refines $\prec$. $\square$

A monomial preorder can be approximated by a weighted preorder by the following lemma, which is well-known in the case of monomial orders (Lemma 2.2).

Lemma 3.3. *Let $\prec$ be a monomial preorder and let $\mathcal{M}$ be a finite set of monomials. Then there exists a weight sequence $\mathbf{w}$ such that the restrictions of $\prec$ and $\prec_{\mathbf{w}}$ to $\mathcal{M}$ coincide.*

Proof. Assume that $\mathcal{M}$ is a set of monomials in s variables $x_1, \ldots, x_s$. We consider the "positive cone"

$$\mathcal{P} := \{\exp(f) - \exp(g) \mid f, g \text{ are monomials such that } g \prec f\} \subseteq \mathbb{Z}^s$$

and the "nullcone"

$$\mathcal{N} := \{\exp(f) - \exp(g) \mid f, g \text{ are incomparable monomials}\} \subseteq \mathbb{Z}^s.$$

We also consider the sets

$$\mathcal{P}^+ := \left\{ \sum_{i=1}^{n} \alpha_i u_i \,\Big|\, n \in \mathbb{N}_{>0},\ u_i \in \mathcal{P},\ \alpha_i \in \mathbb{R}_{>0} \right\} \subseteq \mathbb{R}^s$$

and

$$\mathcal{N}^* := \left\{ \sum_{i=1}^{n} \alpha_i v_i \,\Big|\, n \in \mathbb{N}_{>0},\ v_i \in \mathcal{N},\ \alpha_i \in \mathbb{R} \right\} \subseteq \mathbb{R}^s.$$

Assume that $\mathcal{P}^+ \cap \mathcal{N}^* \neq \emptyset$. Then there exist vectors $u_1, \ldots, u_n \in \mathcal{P}$ and $v_1, \ldots, v_m \in \mathcal{N}$ and real numbers $\alpha_1, \ldots, \alpha_n \in \mathbb{R}_{>0}$ and $\beta_1, \ldots, \beta_m \in \mathbb{R}$ such that

$$\sum_{i=1}^{n} \alpha_i u_i - \sum_{j=1}^{m} \beta_j v_j = 0. \tag{3.1}$$

So $(\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m) \in \mathbb{R}^{n+m}$ is a solution of a system of linear equations with coefficients in $\mathbb{Z}$ that satisfies the additional positivity conditions $\alpha_i > 0$. The existence of a solution in $\mathbb{R}^{n+m}$ satisfying the positivity conditions implies that there also exists a solution in $\mathbb{Q}^{n+m}$ satisfying these conditions. So we may assume $\alpha_i \in \mathbb{Q}_{>0}$ and $\beta_i \in \mathbb{Q}$, and then, by multiplying by a common denominator, $\alpha_i \in \mathbb{N}_{>0}$ and $\beta_i \in \mathbb{Z}$. It follows from the definition of a monomial preorder that $\mathcal{P}$ is closed under addition and that $\mathcal{N}$ is closed under addition and subtraction. Therefore, $\sum_{i=1}^{n} \alpha_i u_i \in \mathcal{P}$ and $\sum_{j=1}^{m} \beta_j v_j \in \mathcal{N}$. Hence (3.1) implies $\mathcal{P} \cap \mathcal{N} \neq \emptyset$. So there exist monomials $g \prec f$ and incomparable monomials h, k such that

$$\exp(f) - \exp(g) = \exp(h) - \exp(k).$$

This implies $fk = gh$. By condition (iv) of the definition of a monomial preorder, gh and gk are incomparable, hence so are fk, gk. This implies that f, g are incomparable, a contradiction. Thus, we must have $\mathcal{P}^+ \cap \mathcal{N}^* = \emptyset$.

Now we form the finite set

$$\mathcal{T} := \{\exp(f) - \exp(g) \mid f, g \in \mathcal{M},\ g \prec f\} \cup \{e_1, \ldots, e_s\},$$

where $e_1, \ldots, e_s \in \mathbb{R}^s$ are the standard basis vectors. Then $\mathcal{T} \subseteq \mathcal{P}$ since $1 \prec x_i$ for all i. We write $T = \{u_1, \ldots, u_n\}$ and form the convex hull

$$\mathcal{H} := \left\{ \sum_{i=1}^{n} \alpha_i u_i \,\Big|\, \alpha_i \in \mathbb{R}_{\geq 0},\ \sum_{i=1}^{n} \alpha_i = 1 \right\} \subseteq \mathcal{P}^+.$$

Since $\mathcal{H}$ is a compact subset of $\mathbb{R}^s$ and $\mathcal{N}^*$ is a linear subspace, there exist $u \in \mathcal{H}$ and $v \in \mathcal{N}^*$ such that the Euclidean distance between u and v is minimal.

Set $w := u - v$. Then

$$w \in (\mathcal{N}^*)^{\perp} \tag{3.2}$$

(the orthogonal complement), since otherwise there would be points in $\mathcal{N}^*$ that are closer to u than v. Set $d := \langle w, w\rangle$, where $\langle\cdot,\cdot\rangle$ denotes the standard scalar product. From $\mathcal{P}^+ \cap \mathcal{N}^* = \emptyset$ we conclude that $d > 0$. Moreover, (3.2) implies $\langle w, u\rangle = \langle w, u - v\rangle = d$. Take $u' \in \mathcal{H}$ arbitrary. Then for every $\alpha \in \mathbb{R}$ with $0 \le \alpha \le 1$ the linear combination $u + \alpha(u' - u)$ also lies in $\mathcal{H}$, so

$$\begin{aligned} d \le \langle u + \alpha(u'-u) - v, u + \alpha(u'-u) - v\rangle &= \langle w + \alpha(u'-u), w + \alpha(u'-u)\rangle \\ &= d + 2\alpha\big(\langle w, u'\rangle - d\big) + \alpha^2\langle u'-u, u'-u\rangle. \end{aligned}$$

Since this holds for arbitrarily small α, we conclude $\langle w, u'\rangle \ge d > 0$. In particular,

$$\langle w, u_i\rangle > 0 \quad \text{for} \quad i = 1, \dots, n. \tag{3.3}$$

Since $\mathcal{N}^*$ has a basis in $\mathbb{Z}^s$, the existence of a vector $w \in \mathbb{R}^s$ satisfying (3.2) and (3.3) implies that there also exists such a vector in $\mathbb{Q}^s$, and then even in $\mathbb{Z}^s$. So we may assume $w \in \mathbb{Z}^s$ and retain (3.2) and (3.3). Since the standard basis vectors e_j occur among the u_i, (3.3) implies that w has positive components.

Let $\mathbf{w} = w_1, w_2, \dots$ be a weight sequence starting with $w_1, \dots, w_s$ chosen above. Let f, g be two arbitrary monomials of $\mathcal{M}$. Then $f \prec_{\mathbf{w}} g$ if and only if $\langle w, \exp(f)\rangle < \langle w, \exp(g)\rangle$. If f and g are incomparable with respect to $\prec$, then $\exp(f) - \exp(g) \in \mathcal{N}^*$, hence $\langle w, \exp(f)\rangle = \langle w, \exp(g)\rangle$ by (3.2). This implies that f and g are incomparable with respect to $\prec_{\mathbf{w}}$. If $f \prec g$, then $\exp(g) - \exp(f) \in \mathcal{T}$, hence $\langle w, \exp(g) - \exp(f)\rangle > 0$ by (3.3). This implies that $f \prec_{\mathbf{w}} g$. So we can conclude that $\prec$ and $\prec_{\mathbf{w}}$ coincide on $\mathcal{M}$. □

Remark. It is clear that any binary relation on the set of monomials satisfying the assertion of Lemma 3.3 is a monomial preorder. Since the lemma is crucial for obtaining the results of this section, this shows that we are working in just the right generality.

Let R be a Noetherian ring and $R[X] := R[x_1, \dots, x_s]$. Let $\prec$ be a monomial preorder. For a polynomial $f \in R[X]$ we define $\mathrm{in}_\prec(f)$ to be the sum of all terms of f that are associated with the minimal monomials appearing in f. As in the previous sections, we call a sequence $a_1, \dots, a_s \in R$ *dependent* with respect to $\prec$ if there exists a polynomial $f \in R[X]$ vanishing at $a_1, \dots, a_s$ such that $\mathrm{in}_\prec(f)$ has at least one invertible coefficient. Otherwise, the sequence is called *independent* with respect to $\prec$. These notions cover both weighted (in-)dependent sequences and (in-)dependent sequences with respect to a monomial order.

The following result allows us to reduce the study of these notions to weighted independent sequences and dependent sequences with respect to a monomial order.

Proposition 3.4. *Let $a_1, \dots, a_s \in R$ be a sequence of elements.*

(a) *The sequence is independent with respect to every monomial preorder if it is weighted independent with respect to every weight sequence.*
(b) *The sequence is dependent with respect to every monomial preorder if it is dependent with respect to every monomial order.*

Proof. (a) Assume that $a_1, \dots, a_s$ is weighted independent with respect to every weight sequence. If $a_1, \dots, a_s$ is dependent with respect to some monomial preorder $\prec$, there is a polynomial $f \in R[X]$ vanishing at $a_1, \dots, a_s$ such that $\mathrm{in}_\prec(f)$ has an invertible coefficient. By Lemma 3.3 there exists a weight sequence $\mathbf{w}$ such that $\mathrm{in}_\prec(f) = \mathrm{in}_{\mathbf{w}}(f)$. So $a_1, \dots, a_s$ is weighted dependent with respect to $\mathbf{w}$, a contradiction.

(b) Assume that $a_1, \dots, a_s$ is dependent with respect to every monomial order. If $a_1, \dots, a_s$ is independent with respect to some monomial preorder $\prec$, we use Lemma 3.2

to find a monomial order $\prec^*$ that refines $\prec$. If $f \in R[X]$ is a polynomial vanishing at $a_1, \dots, a_s$, then $\text{in}_\prec(f)$ has no invertible coefficient. Since the least term $\text{in}_{\prec^*}(f)$ of f is minimal with respect to $\prec^*$, is is also minimal with respect to $\prec$, so it is a term of $\text{in}_\prec(f)$. Therefore the coefficient of $\text{in}_{\prec^*}(f)$ is not invertible. But this means that the sequence is independent with respect to $\prec^*$, a contradiction. $\square$

Combining Proposition 3.4 with Theorems 1.8(a) and 2.7(b), we obtain the following generalization of the main results of the two previous sections.

Theorem 3.5. *Let R be a Noetherian ring and s a positive integer.*

(a) *If $s \le \dim R$, there exists a sequence $a_1, \dots, a_s \in R$ that is independent with respect to every monomial preorder.*
(b) *If $s > \dim R$, every sequence $a_1, \dots, a_s \in R$ is dependent with respect to every monomial preorder.*

As a consequence, $\dim R$ is the supremum of the length of independent sequences with respect to an arbitrary monomial preorder.

4. Algebras over a Jacobson ring

In this section we extend our investigation to algebras over a ring. Our aim is to generalize the characterization of the Krull dimension of algebras over a field by means of the transcendence degree.

Let A be an algebra over a ring R. Given a monomial preorder $\prec$, we say that a sequence $a_1, \dots, a_s$ of elements of A is *dependent over* R with respect to $\prec$ if there exists a polynomial $f \in R[X] := R[x_1, \dots, x_s]$ vanishing at $a_1, \dots, a_s$ such that $\text{in}_\prec(f)$ has at least one coefficient that is invertible in R. Otherwise, the sequence is called *independent over* R with respect to $\prec$. Note that if R is a field, these are just the usual notions of algebraic dependence and independence, and they do not depend on the choice of the monomial preorder. In this case, it is well known that $\dim A$ is equal to the transcendence degree of A over R. So we may ask whether $\dim A$ is equal to the maximal length of independent sequences over R with respect to $\prec$.

The following example shows that this question has a negative answer in general.

Example 4.1. Let R be an one-dimensional local domain. Let $A = R[a^{-1}]$, where $a \neq 0$ is an element in the maximal ideal of R. Then $\dim A = 0$, whereas a is an independent element over R with respect to every monomial preorder. (In fact, there exists only one monomial preorder in just one variable.)

We shall see that the above question has a positive answer if R is a Noetherian Jacobson ring. Recall that R is called a *Jacobson ring* (or Hilbert ring) if every prime of R is the intersection of maximal ideals. It is well known that every finitely generated algebra over a field is a Jacobson ring (see Eisenbud [9, Theorem 4.19]). More examples are given by tensor products of extensions of a field with finite transcendence degree [16].

Clearly, R is a Jacobson ring if and only if every nonmaximal prime P of R is the intersection of primes $P' \supset P$ with $\text{ht}(P'/P) = 1$. Therefore, the following lemma will be useful in studying Jacobson rings. This lemma seems to be folklore. Since we could not find any references in the literature, we provide a proof for the convenience of the reader.

Lemma 4.2. *Let R be a Noetherian ring and P a nonmaximal prime of R.*

(a) *For every prime $Q \supset P$ with $\text{ht}(Q/P) \ge 2$, there exist infinitely many primes P' with $P \subseteq P' \subseteq Q$ and $\text{ht}(P'/P) = 1$ in Q.*

(b) *If $\mathcal{M}$ is a set of primes $P' \supset P$ with* $\mathrm{ht}(P'/P) = 1$*, then* $P = \cap_{P' \in \mathcal{M}} P'$ *if and only if $\mathcal{M}$ is infinite.*

Proof. (a) By factoring out P and localizing at Q we may assume that P is the zero ideal of a local domain R with maximal ideal Q. We have to show that the set of height one primes of R is infinite. By Krull's principal theorem, every element $a \neq 0$ in Q is contained in some height one prime P'. So Q is contained in the union of all height one primes of R. If the number of these primes were finite, it would follow by the prime avoidance lemma that Q is contained in one of them, contradicting the hypothesis $\mathrm{ht}(Q) \geq 2$.

(b) Let $I = \cap_{P' \in \mathcal{M}} P'$. If $P \neq I$, every prime P' of $\mathcal{M}$ is a minimal prime over I. Hence $\mathcal{M}$ is finite because R is Noetherian. Conversely, if $\mathcal{M}$ is finite, then $\mathcal{M}$ is the set of minimal primes over I. This implies $\mathrm{ht}(P) < \mathrm{ht}(I)$, hence $P \neq I$. $\square$

Corollary 4.3. *A Noetherian ring R is a Jacobson ring if and only if for every prime P with* $\dim R/P = 1$ *there exist infinitely many maximal ideals containing P.*

Proof. By Lemma 4.2, every prime P of a Noetherian ring R with $\dim R/P \geq 2$ is the intersection of primes $P' \supset P$ with $\mathrm{ht}(P'/P) = 1$. Therefore, R is a Jacobson ring if and only if every prime P with $\dim R/P = 1$ is the intersection of maximal primes. By Lemma 4.2(b), this is equivalent to the condition that there exist infinitely many maximal ideals containing P. $\square$

We use the above results to prove the following lemma which will play a crucial role in our investigation on independent sequences over R.

Lemma 4.4. *Let a be an element of a Noetherian ring R and set*

$$U_a := \{a^n(1 + ax) \mid n \in \mathbb{N}_0,\ x \in R\}.$$

Then the localization $U_a^{-1}R$ is a Jacobson ring.

Proof. We will use the inclusion-preserving bijection between the primes of $S := U_a^{-1}R$ and the primes P of R satisfying $U_a \cap P = \emptyset$. Let P be such a prime of R with $\dim\left(S/U_a^{-1}P\right) = 1$. Then there exists a prime $P_1 \supset P$ of R with $\mathrm{ht}(P_1/P) = 1$ and $U_a \cap P_1 = \emptyset$. The latter condition implies $a \notin P_1$ and $1 \notin (P_1, a)$. Let Q be a prime of R containing (P_1, a). Then $\mathrm{ht}(Q/P) \geq 2$. By Lemma 4.2(a), the set

$$\mathcal{M} := \left\{P' \in \mathrm{Spec}(R) \mid P \subset P' \subset Q,\ \mathrm{ht}(P'/P) = 1\right\}$$

is infinite. Consider the set $\mathcal{N} := \{P' \in \mathcal{M} \mid U_a \cap P' \neq \emptyset\}$. If $\mathcal{N}$ is infinite, $P = \bigcap_{P' \in \mathcal{N}} P'$ by Lemma 4.2(b). Since $U_a \cap P = \emptyset$, $a \notin P$. Therefore, there exists a prime $P' \in \mathcal{N}$ such that $a \notin P'$. Since $U_a \cap P' \neq \emptyset$, this implies $1 + ax \in P'$ for some $x \in R$. Hence $1 \in (P', a) \subseteq Q$, a contradiction. So $\mathcal{N}$ must be finite, and we can conclude that $\mathcal{M} \setminus \mathcal{N}$ is infinite. By the definition of $\mathcal{M}$ and $\mathcal{N}$, the set of primes $P' \supset P$ with $\mathrm{ht}(P'/P) = 1$ and $U_a \cap P' = \emptyset$ is infinite. Since this set corresponds to the set of maximal ideals of S containing $U_a^{-1}P$, the assertion follows from Corollary 4.3. $\square$

Remark. The localization $U_a^{-1}R$ from Lemma 4.4 was already used by Coquand and Lombardi to give a short proof for the fact that the Krull dimension of a polynomial ring over a field is equal to the number of variables [5]. They called it the boundary of a in R.

Now we are going to give a characterization of the Krull dimension of algebras over a Jacobson ring R by means of independent elements over R with respect to an arbitrary monomial preorder $\prec$. First we need to consider the case where $\prec$ is the lexicographic order with $x_i > x_{i+1}$ for all i.

We call an R-algebra *subfinite* if it is a subalgebra of a finitely generated R-algebra. A subfinite algebra needs not to be finitely generated.

Theorem 4.5. *Let A be a subfinite algebra over a Noetherian Jacobson ring R and let s be a positive integer. There exists a sequence $a_1, \dots, a_s \in A$ that is independent over R with respect to the lexicographic order if and only if $s \le \dim A$.*

Proof. If $s \le \dim A$, Lombardi [13] (which does require A to be Noetherian) tells us that there exists a sequence of length s that is independent over A with respect to the lexicographic order. Therefore it is also independent over R.

The next step is to prove the converse under the hypothesis that A is finitely generated. We use induction on s. We may assume that $A \ne \{0\}$, $\dim A < \infty$, and $s = \dim A + 1$. We have to show that every sequence $a_1, \dots, a_s \in A$ is dependent over R with respect to the lexicographic order.

Let T be the set of univariate polynomials $f \in R[x]$ whose initial term $\mathrm{in}(f)$ has coefficient 1. Since T is multiplicatively closed, so is the set

$$U := \{f(a_s) \mid f \in T\} \subseteq A.$$

Let $A' := U^{-1}A$. If $\dim A' = s - 1$, then A has a height $s-1$ prime P with $U \cap P = \emptyset$. This prime must be maximal because $\dim A = s - 1$. Since R is a Jacobson ring, A/P is a finite field extension of $R/(R \cap P)$ [9, Theorem 4.19]. Since $U \cap P = \emptyset$, $a_s \notin P$. These facts imply that there exists $g \in R[x]$ such that $a_s g(a_s) - 1 \in P$. But $1 - xg \in T$, so $1 - a_s g(a_s) \in U \cap P$, a contradiction. So we can conclude that $\dim A' < s - 1$.

If $A' = \{0\}$ (which must happen if $s = 1$), then $0 \in U$, hence there exists $f \in T$ with $f(a_s) = 0$. So the sequence $a_1, \dots, a_s$ is dependent over R with respect to the lexicographic order. Having dealt with this case, we may assume $A' \ne \{0\}$. Let $R' := U^{-1}R[a_s]$. Then A' is finitely generated as an R'-algebra. By Lemma 4.4, R' is a Jacobson ring. So we may apply the induction hypothesis to A'. This tells us that the sequence $a_1, \dots, a_{s-1}$ (as elements of A') is dependent over R' with respect to the lexicographic order. Thus, there exists a polynomial $g \in R'[x_1, \dots, x_{s-1}]$ vanishing at $a_1, \dots, a_{s-1}$ such that the coefficient of $\mathrm{in}_{\mathrm{lex}}(g)$ is invertible in R'. We may assume that this coefficient is 1. By the definition of A' there exists $c_0 \in R$ such that $c_0 g \in R[a_s][x_1, \dots, x_{s-1}]$ and $(c_0 g)(a_1, \dots, a_{s-1}) = 0$ (as an element of A). Replacing every coefficient $c \in R[a_s]$ of the polynomial $c_0 g$ by a polynomial $c^* \in R[x_s]$ with $c^*(a_s) = c$, we obtain a polynomial $g^* \in R[x_1, \dots, x_s]$ vanishing at $a_1, \dots, a_s$. Since $c_0 \in U$, we may choose $c_0^* \in T$. Clearly, the coefficient of $\mathrm{in}_{\mathrm{lex}}(g^*)$ is equal to the coefficient of $\mathrm{in}(c_0^*)$, which is 1. This shows that $a_1, \dots, a_s$ are dependent over R with respect to the lexicographic order.

Now we deal with the case A is a subalgebra of a finitely generated R-algebra B. Let $P_1, \dots, P_n \in \mathrm{Spec}(B)$ be the minimal primes of B, and assume that we can show that for every i, the images of $a_1, \dots, a_s$ in $A/(A \cap P_i)$ are dependent over R with respect to the lexicographic order. Then for every i, there exists a polynomial $f_i \in R[x_1, \dots, x_s]$ with $f_i(a_1, \dots, a_s) \in P_i$ such that the coefficient of $\mathrm{in}_{\mathrm{lex}}(f_i)$ is invertible. Since $\prod_{i=1}^n f_i(a_1, \dots, a_s)$ lies in the nilradical of B, there exists k such the polynomial $f := \prod_{i=1}^n f_i^k$ vanishes at $a_1, \dots, a_s$. Since the coefficient of $\mathrm{in}_{\mathrm{lex}}(f)$ is also invertible, this shows that $a_1, \dots, a_s$ are dependent over R with respect to the lexicographic order. So we may assume that B is an integral domain. By Giral [10, Proposition 2.1(b)] (or [11, Exercise 10.3]), there exists a nonzero $a \in A$ such that $A[a^{-1}]$ is a finitely generated R-algebra. Since $\dim A[a^{-1}] \le \dim A < s$, the sequence $a_1, \dots, a_s$ is dependent over R with respect to the lexicographic order. This completes the proof. ☐

One can use Theorem 4.5 to prove the existence of nontrivial relations between algebraic numbers (i.e., elements of an algebraic closure of $\mathbb{Q}$).

Example 4.6. Let a and b be two nonzero algebraic numbers. There exists $d \in \mathbb{Z} \setminus \{0\}$ such that a and b are integral over $\mathbb{Z}[d^{-1}]$. So $A := \mathbb{Z}\left[a, b, d^{-1}\right]$ has Krull dimension 1. By Theorem 4.5, there is a polynomial $f \in \mathbb{Z}[x_1, x_2]$ vanishing at a, b such that the coefficient of $\mathrm{in}_{\mathrm{lex}}(f)$ is 1. Let $\mathrm{in}_{\mathrm{lex}}(f) = x_1^m x_2^n$. Then all monomials of f are divisible by x_1^m. Hence we may assume $m = 0$. Thus,

$$b^n = a \cdot g(a, b) + b^{n+1} \cdot h(a, b)$$

for some $g, h \in \mathbb{Z}[x_1, x_2]$. It is not clear how the existence of such a relation follows directly from properties of algebraic numbers. In the case that $\mathbb{Z}[a, b]$ is a Dedekind ring, we derived such a relation in Example 2.1.

To the best of our knowledge, the following immediate consequence of Theorem 4.5 is new even for finitely generated algebras.

Corollary 4.7. *Let R be a Noetherian Jacobson ring. If $A \subset B$ are subfinite R-algebras, then*

$$\dim A \leq \dim B.$$

Now we turn to arbitrary monomial preorders and prove the main result of this section. The proof relies on Corollary 4.7.

Theorem 4.8. *Let A be a subfinite algebra over a Noetherian Jacobson ring R and let s be a positive integer.*

(a) *If $s \leq \dim A$, there exists a sequence $a_1, \dots, a_s \in A$ that is independent over R with respect to every monomial preorder.*
(b) *If $s > \dim A$, every sequence $a_1, \dots, a_s \in A$ is dependent over R with respect to every monomial preorder.*

Proof. (a) Let A be a subalgebra of a finitely generated R-algebra B. Since $\sqrt{0_A} = \sqrt{0_B} \cap A$ and since the nilradical is the intersection of all minimal primes, the set of the minimal primes of A is contained in the set of prime ideals of the form $P \cap A$, where P is a minimal prime ideal of B. Therefore, there exists a minimal prime ideal P of B such that $\dim A/P \cap A = \dim A$. If $a_1, ..., a_s \in A$ is independent over R in $A/P \cap A$, then it is also independent over R in A. Therefore, we may replace A by $A/P \cap A$ and assume that B is an integral domain. By Giral [10, Proposition 2.1(b)] (or [11, Exercise 10.3]), there exists a nonzero $a \in A$ such that $A[a^{-1}]$ is a finitely generated R-algebra. Then $\dim A[a^{-1}] \leq \dim A$. By Corollary 4.7, $\dim A \leq \dim A[a^{-1}]$. Hence $s \leq \dim A = \dim A[a^{-1}]$. Choose $a_1, ..., a_s \in A$ such that $(a_1, ..., a_s)A[a^{-1}]$ has a minimal prime ideal of height s. Then $a_1, ..., a_s$ is independent over R with respect to every monomial preorder by Proposition 1.7 and Theorem 3.4(a).

(b) Let $A' := R[a_1, \dots, a_s] \subseteq A$. By Corollary 4.7, $\dim A' \leq \dim A$. So Theorem 3.5(b) yields for every monomial preorder $\prec$ a polynomial $f \in A'[x_1, \dots, x_s]$ vanishing at $a_1, \dots, a_s$ such that $\mathrm{in}_\prec(f)$ has an invertible coefficient c_0. We may assume $c_0 = 1$. Write $f = \sum_{i=0}^n c_i t_i$ with $c_i \in A'$ and t_i pairwise different monomials such that t_0 is minimal among the t_i. Choose polynomials $c_i^* \in R[x_1, \dots, x_s]$ with $c_i^*(a_1, \dots, a_s) = c_i$ and $c_0^* = 1$. Set set $f^* = \sum_{i=0}^n c_i^* t_i$. Then f^* is a polynomial of $R[x_1, \dots, x_s]$ vanishing at $a_1, \dots, a_s$. From the compatibility of monomial preorders with multiplication we conclude that t_0 is a term of $\mathrm{in}_\prec(f^*)$. This shows that the sequence $a_1, \dots, a_s$ is dependent over R with respect to $\prec$. $\square$

Theorem 4.8 generalizes a result of Giral [10] which says that the dimension of a subfinite algebra over a field is equal to its transcendence degree.

As a consequence of the above results, we give a characterization of Jacobson rings, which implies that the hypothesis that R is a Jacobson ring cannot be dropped from Corollary 4.7 and Theorem 4.8.

Corollary 4.9. *For a Noetherian ring R, the following statements are equivalent:*

(a) *R is a Jacobson ring.*
(b) *For every subfinite R-algebra A and every monomial preorder,* $\dim A$ *is the supremum of the length of independent sequences over R in A.*
(c) *If $A \subseteq B$ is a pair of subfinite R-algebras, then* $\dim A \leq \dim B$.

Proof. The only implication that requires a proof is that (c) implies (a). But if R is not a Jacobson ring, then by [9, Lemma 4.20], R has a nonmaximal prime ideal P such that $A := R/P$ contains a nonzero element b for which $B := A[b^{-1}]$ is a field. So (c) fails to hold. □

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