A1) $\operatorname{Spec} R/N \longrightarrow \operatorname{Spec} R$

$\bar{p} \longmapsto q^{-1}(\bar{p})$

$q(p) = p/N \longleftrightarrow p$ ← welldef. bc. $N \subseteq p$ $\forall p$ and q surj.

Then $\bar{p} = q q^{-1}(\bar{p})$ because q surj.

$p = q^{-1} q(p) = p + N$ because $N \subseteq p$.

$\Rightarrow q^{\#}$ bijective

Claim: $\forall f \in R : \operatorname{Spec}(R/N) \supseteq D(q(f)) = D(f) \subseteq \operatorname{Spec}(R)$

Proof: If $\bar{p} \in D(q(f)) \Leftrightarrow q(f) \notin \bar{p}$

$\Longleftrightarrow f \in q^{-1}(\bar{p}) = p \Leftrightarrow p \in D(f)$.

A2: Consider $0 \to I \longrightarrow k[x_1, \dots, x_n] \xrightarrow{f} R \to 0$ (for some f).

And $X = Z(I) \subseteq k^n$.

Then by def: $k[X] = k[x_1, \dots, x_n]/I \cong R$,

A3) 1)

$$\begin{tikzcd}
0 \arrow[r] & \ker\varphi_U \arrow[r] \arrow[d] \arrow[dd, bend right] & F(U) \arrow[r, "\varphi_U"] \arrow[d] \arrow[dd, bend right] & G(U) \arrow[d] \arrow[dd, bend left] \\
0 \arrow[r] & \ker\varphi_V \arrow[r] \arrow[d] & F(V) \arrow[r, "\varphi_V"] \arrow[d] & G(V) \arrow[d] \\
0 \arrow[r] & \ker(\varphi_x) \arrow[r] & F_x \arrow[r, "\varphi_x"] & G_x
\end{tikzcd}$$

Claim: $\ker(\varphi_x)$ has the univ. property of $\underrightarrow{\lim}(\ker \varphi_U)$!

i) Konstruction of $\ker \varphi_U \xrightarrow{\alpha_U} \ker(\varphi_x)$:

let $s_U \in F(U)$ s.th. $\varphi_U(s_U) = 0$ in $G(U)$

Then $\varphi_x((s_U)_x) = (\varphi_U(s_U))_x = 0_x = 0$ in G_x

$\Rightarrow (s_U)_x \in \ker(\varphi_x)$

$\rightsquigarrow \ker \varphi_U \xrightarrow{\alpha_U} \ker(\varphi_x)$, $s_U \longmapsto (s_U)_x$.

ii) $$\begin{tikzcd}
\ker\varphi_U \arrow[r] \arrow[dr, "\alpha_U"'] & \ker\varphi_V \arrow[d, "\alpha_V"] \\
& \ker\varphi_x
\end{tikzcd}$$ obvious.

Consider now any ab. group A with $g_U : \ker \varphi_U \to A$ $\forall U$ which are compatible with restrictions.

iii) Uniqueness of $h : \ker(\varphi_x) \longrightarrow A$.

Let $s \in \ker(\varphi_x)$, i.e. $s \in \mathcal{F}_x$ with $\varphi_x(s) = 0$. Then represent s by an element $s_U \in \mathcal{F}_U$. Then $(\varphi_U(s_U))_x = \varphi_x((s_U)_x) = \varphi_x(s) = 0$.

Hence $\exists$ $V \subseteq U$ s.th. $\varphi_U(s_U)|_V = 0$ in $\mathcal{G}(V)$

$\Rightarrow$ $s_U|_V \in \ker(\varphi_V)$ another representative of s.

But then: $h(s) = h(\alpha_V(s_U|_V)) = g_V(s_U|_V) \in A$

$\Rightarrow$ $h(s)$ uniquely determined!

ii) Existence of $h : \ker(\varphi_x) \to A$.

Define $h : \ker(\varphi_x) \to A$ by

$$h(s) = g_V(s_V) \quad \text{for any repr. } s_V \in \ker(\varphi_V).$$

Claim: h well-def!.

Proof:

Assume we have two repr. s_V and $s_{V'}$ of s. Then there is $W \subseteq V \cap V'$ s.th. $s_V|_W = s_{V'}|_W \in \ker \varphi_W$. Now:

$$g_V(s_V) \overset{\circledast}{=} g_W(s_V|_W) = g_W(s_{V'}|_W) \overset{\circledast}{=} g_{V'}(s_{V'})$$ $\circledast$ compat. of g_U with restriction.

$\Rightarrow$ $h(s)$ independent of $s_V, s_{V'}$.

Similarly $(\operatorname{im} \varphi_x) = (\operatorname{preim} \varphi)_x \cong (\operatorname{im} \varphi)_x$.

(2) injective: We have a morphism $\underline{0} \longrightarrow \ker(\varphi)$.

$$\varphi \text{ injective} \iff \underline{0} \xrightarrow{\cong} \ker(\varphi) \iff (\underline{0})_x \xrightarrow{\cong} (\ker \varphi)_x \ \forall x$$

$$\overset{a)}{\iff} 0 = \ker(\varphi_x) \ \forall x \iff \varphi_x \text{ inj. } \forall x.$$

surj. We have $\operatorname{pre-im} \varphi \longrightarrow \mathcal{G}$

By univ. prop. $\operatorname{im}\varphi = \operatorname{pre\,im} \varphi^+ \longrightarrow \mathcal{G}$.

$$\varphi \text{ surj.} \iff \operatorname{im}\varphi \xrightarrow{\cong} \mathcal{G} \iff (\operatorname{im}\varphi)_x \xrightarrow{\cong} \mathcal{G}_x \ \forall x$$

$$\iff (\operatorname{im}\varphi_x) \xrightarrow{\cong} \mathcal{G}_x \ \forall x \iff \varphi_x \text{ surj. } \forall x.$$

4)(1) Sei $\mathcal{F}(\emptyset)=0$. ~~Dan~~ ..

i) $\varrho_{UU} = id_{\mathcal{F}(U)}$ by def.

ii) $\varrho_{VW}\circ\varrho_{UV} = id_A \circ id_A = id_A = \varrho_{UW}$ if $W \neq \emptyset$

$\varrho_{VW}\circ\varrho_{UV} = 0 = \varrho_{UW}$ if $W = \emptyset$.

(2). $X = \{x, y\}$ discrete, $A = \mathbb{Z}$.

Cover $X = \{x\} \cup \{y\}$. Take $s_{\{x\}} \equiv 0$, $s_{\{y\}} \equiv 1$.

Then $s_{\{x\}}|_{\{x\}\cap\{y\}} = 0 = s_{\{y\}}|_{\dots}$ as $\{x\}\cap\{y\} = \emptyset$.

But $\nexists\, s \in \mathcal{F}(X) = A$ with $s|_{\{x\}} = 0$ and $s|_{\{y\}} = 1$.

(3) Let $U = \bigcup_{i\in I} U_i \supseteq V = \bigcup_{j\in J} V_j$ be decomp. into conn. comp.

Then $\forall j \in J\ \exists!\, i \in I : V_j \subseteq U_i$. (by connectedness of j.)

This defines a map $\tilde{\varrho}_{UV} : J \longrightarrow I$.

$\rightsquigarrow \varrho_{UV} : \mathcal{G}(U) = \prod \mathcal{G}(U_i) \longrightarrow \mathcal{G}(V) = \prod \mathcal{G}(V_j)$

$(\underset{\in A}{s_i}) \longmapsto (\underset{\in A}{t_j})$ with $t_j = s_i$ iff $\tilde{\varrho}_{UV}(j) = i$.

α) $\varrho_{VW}\circ\varrho_{UV} = \varrho_{UW}$: $W = \bigcup_{k\in K} W_k \subseteq V = \bigcup_{j\in J} V_j \subseteq U = \bigcup_{i\in I} U_i$

$\forall k \in K\ \exists!\, j \in J : W_k \subseteq V_j$ $\quad \exists!\, i \in I : V_j \subseteq U_i$

This is the unique i with $W_k \subseteq U_i$ $\Rightarrow$ $\tilde{\varrho}_{UW} = \tilde{\varrho}_{UV} \circ \tilde{\varrho}_{VW}$

Then by def. $\varrho_{UW} = \varrho_{VW}\varrho_{UV}$

β) Let $U \subseteq X$ open, $\{U^{(\alpha)}\}$ open cover of U.

Let $s \in \mathcal{G}(U)$ with $s|_{U^{(\alpha)}} = 0\ \forall U^{(\alpha)}$ in the open cover.

Let $U = \bigcup U_j$ be the connected components and $s = (s_j) \in \prod \mathcal{G}(U_i)$.

~~Fix~~ Choose any i and any $U^{(\alpha)}$ with $U^{(\alpha)} \cap U^{(i)} \neq 0$. Let V be any conn. comp. of $U^{(\alpha)} \cap U^{(i)}$

$$\begin{tikzcd}
s \arrow[rrr, mapsto] \arrow[ddd, mapsto] & & & 0 \arrow[ddd, mapsto] \\
& \mathcal{G}(U) \arrow[r] \arrow[d] & \mathcal{G}(U^{(\alpha)}) \arrow[d] & \\
& \mathcal{G}(U_i) \arrow[r] & \mathcal{G}(V) & \\
s_i \arrow[rrr, mapsto] & & & 0
\end{tikzcd}$$

But if $V \subseteq U_i$ open and V and U_i connected, then by def

$\varrho_{U_iV} = id_A : \mathcal{G}(U_i) \cong A \to \mathcal{G}(V) \cong A$.

$\Rightarrow s_i = 0$

$\Rightarrow s = (s_i) = (0) = 0 \in \mathcal{G}(U)$.

β) Let $U \subseteq X$ open, $\{U^{(n)}\}$ cover of U and $s^{(n)} \in \mathcal{G}(U^{(n)})$ this s.th.

$s^{(n)}|_{U^{(n)} \cap U^{(m)}} = s^{(m)}|_{U^{(n)} \cap U^{(m)}}$. $\forall n, m$.

Let $U_i \subseteq U$ one connected component.

Claim 1

$\exists\, a_{U_i} \in A \mid s^{(n)}|_{U_i \cap U^{(n)}} = \cancel{\text{const}}\ a_{U_i}$ (i.e. a_{U_i} on every conn. comp. of $U_i \cap U^{(n)}$).

Proof of Claim 1:

Let $S = \{(U^{(n)}, \text{ conn. comp. } U_j^{(n)} \subseteq U^{(n)} \cap U^{(n)})\}$. ~~Let T=~~ Choose any $U_0^{(n)}$ and $U_{j,0}^{(n)} \subseteq U_0^{(n)}$. Then define $a := s^{(n)}|_{U_{j,0}^{(n)}} \in \mathcal{G}(U_{j,0}^{(n)}) = A$.

and $T = \{(U^{(n)}, U_j^{(n)}) \mid s^{(n)}|_{U_j^{(n)}} = a \in \mathcal{G}(U_j^{(n)}) = A\} \subseteq S$.

Assume $T \neq S$. Then

$$V = \bigcup_{(U^{(n)}, U_j^{(n)}) \in T} U_j^{(n)} \qquad \text{and} \qquad W = \bigcup_{(U^{(n)}, U_j^{(n)}) \notin T} U_j^{(n)}$$

are open in U and cover it. Furthermore if $x \in V \cap W$ choose

$(U^{(n)}, U_j^{(n)}) \in T$ and $(U^{(m)}, U_{j'}^{(m)}) \notin T$ mit $x \in U_j^{(n)}, U_{j'}^{(m)}$.

Then $a = s^{(n)}|_{\underbrace{U_j^{(n)} \cap U_{j'}^{(m)}}_{\text{non-empty!}}} = s^{(m)}|_{U_j^{(n)} \cap U_{j'}^{(m)}} \neq a$. ↯

$\Rightarrow T = S \Rightarrow$ Claim.

Define now $s = (a_{U_i}) \in \mathcal{G}(U) = \prod \mathcal{G}(U_i)$.

Claim 2

$s|_{U^{(n)}} = s^{(n)}$

Proof:

By def $s|_{U_i \cap U^{(n)}} = a_{U_i} \overset{\text{Claim 1.}}{=} s^{(n)}|_{U_i \cap U^{(n)}}$ But $\{U_i\}$ $U^{(n)}$ cover of $U^{(n)} \to \checkmark$.

(4): We check the universal property for

$$\mathcal{F} \xrightarrow{\vartheta} \mathcal{G}, \quad \mathcal{F}(U) \longrightarrow \mathcal{G}(U) = \prod \mathcal{G}(U_i), \quad a \longmapsto (a)_i .$$

Let $\mathcal{H}$ be any sheaf on X and $f: \mathcal{F} \to \mathcal{H}$ morphism of presheaves.

Claim 1: There is at most one $g: \mathcal{G} \to \mathcal{H}$ with $f = g \circ \vartheta$.

Proof:

For U connected we have $\vartheta: \mathcal{F}(U) \xrightarrow{\cong} \mathcal{G}(U)$ iso, hence we have to define $\underset{\mathcal{G}(U) = A = \mathcal{F}(U)}{g(a) = f(a)}$. If U is not connected, write $U = \bigcup U_i$. Let $s(a_i) \in \mathcal{G}(U) = \prod \mathcal{G}(U_i)$. Then ~~consider~~ as U_i connected

$$g(s)|_{U_i} = g(s|_{U_i}) = g(a_i) = f(a_i).$$

But $g(s)$ is uniquely determined by its restriction to the open cover $\{U_i\}$. $\Rightarrow g(s)$ is unique.

Claim 2: There is such a g.

Proof.

Let $U = \bigcup U_i$ decomp. into conn. comp. Then define

$$\begin{array}{ccccccc} \mathcal{G}(U) & = & \prod \mathcal{G}(U_i) & \longrightarrow & \prod \mathcal{H}(U_i) & \overset{\cong ?}{\longleftrightarrow} & \mathcal{H}(U) \\ s & = & (a_i) & \longrightarrow & (f(a_i)) & \longleftarrow & g(s) \end{array}$$

unique preimage.

The $f(a_i) \in \mathcal{H}(U_i)$ satisfy $f(a_i)|_{U_i \cap U_j} = f(a_j)|_{U_i \cap U_j}$ as $U_i \cap U_j = \emptyset \Rightarrow$ We may glue ~~all~~ $\{f(a_i)\}$ to some section $g(s) \in \mathcal{H}(U)$.

Claim 3: g is additive

Claim 4: g is compatible with restrictions.

Let $V = \bigcup V_j \subset U = \bigcup U_j$ and $\tilde{\varrho}_{UV}: J \to I$ as in (3). Then

$$\begin{array}{ccccccc} s & & (a_i) & & (f(a_i)) & & g(s) \\ \mathcal{G}(U) & = & \prod \mathcal{G}(U_i) & \longrightarrow & \prod \mathcal{H}(U_i) & \longleftarrow & \mathcal{H}(U) \\ \downarrow & & \downarrow & & & & \downarrow \\ s|_V & & (a_{\tilde{\varrho}(j)}) & & (f(a_{\tilde{\varrho}(j)})) & & g(s|_V) \\ \mathcal{G}(V) & = & \prod \mathcal{G}(V_j) & \longrightarrow & \prod \mathcal{H}(V_j) & \longleftarrow & \mathcal{H}(V) \end{array}$$

with $g(s)|_{V_j} = (g(s)|_{U_{\tilde{\varrho}(j)}})|_{V_j} = f(a_{\tilde{\varrho}(j)})|_{V_j} = g(s|_V)|_{V_j} \Rightarrow \checkmark$.

Alternative (und ~~kürzer~~ shorter:)

Consider ~~Betrachte~~ $\vartheta : \mathcal{F} \to \mathcal{G}$.

Show: ~~Dann zeige~~: $\vartheta_x = id_A : \mathcal{F}_x = A \longrightarrow \mathcal{G}_x = A$.

~~Dann betrachte~~ Consider:

$$\begin{tikzcd} \mathcal{F} \arrow[r] \arrow[dr] & \mathcal{F}^+ \arrow[d, "\vartheta^+"'] \\ & \mathcal{G} \end{tikzcd} \quad \longleftarrow \exists \text{ ~~nach univ. Eig~~ univ. Prop.}$$

$\Rightarrow \vartheta^+$ ist Morphismus of Sheaves, and Iso ~~auf allen Halmen~~ on Stalks

$\Rightarrow \vartheta^+$ ist Iso.