Ex 1:

For $R = k[x,y]/(x^2+y^3)$:

$$\Omega_{R/k} = \operatorname{coker}\left((x^2+y^3) \xrightarrow{d} \Omega_{k[x,y]/k} \otimes R \right) = Rdx \oplus Rdy \Big/ d(x^2+y^3) =$$
$$= Rdx \oplus Rdy \Big/ 2x\,dx + 3y\,dy.$$

On $D(xy) \subseteq \operatorname{Spec} R$ we have $\Omega_{R/k} \otimes_R R[\frac{1}{x}, \frac{1}{y}] \cong R[\frac{1}{x}, \frac{1}{y}]$ ~~locally~~ free of rank 1 (as either $2x \in R[\frac{1}{x}, \frac{1}{y}]^\times$ or $3y \in R[\frac{1}{x}, \frac{1}{y}]^\times$, slightly depending on $\operatorname{char}(k)$).

Furthermore we have

$$\Omega_{R/k} \otimes_R R/(x,y) \cong R/(x,y)\,dx \oplus R/(x,y)\,dy \Big/ (2x\,dx + 3y\,dy)\cdot R/(x,y) =$$
$$= R/(x,y)\,dx \oplus R/(x,y)\,dy$$

free of rank 2.

But if $\Omega_{R/k}$ is locally free the value $\operatorname{rk}_{\kappa(p)}(\Omega_{R/k} \otimes_R \kappa(p))$ has to be constant for all $p \in \operatorname{Spec} R$ (where $\kappa(p)$ denotes the residue field). This is not true as the computation above show.

For $R = k[x,y,z]/(z^2 - xy)$:

$$\Omega_{R/k} = \operatorname{coker}\left((z^2 - xy) \xrightarrow{d} \Omega_{k[x,y,z]/k} \otimes R \right) = Rdx \oplus Rdy \oplus Rdz \Big/ d(z^2 - xy) =$$
$$= Rdx \oplus Rdy \oplus Rdz \Big/ 2z\,dz - y\,dx - x\,dy.$$

On $D(x) \subseteq \operatorname{Spec} R$ we have

$$\Omega_{R/k} \otimes_R R[\tfrac{1}{x}] \cong R[\tfrac{1}{x}]\,dy \oplus R[\tfrac{1}{x}]\,dz$$

free of rank 2 (and similarly on $D(y)$).

But $\Omega_{R/k} \otimes_R R/(x,y,z) \cong R/(x,y,z)\,dx \oplus R/(x,y,z)\,dy \oplus R/(x,y,z)\,dz$ free of rank 3.

Thus $\Omega_{R/k}$ can not be locally free, either..

Ex 2.8

(1): Let $x \in K$ be a primitive element. Then $K = k(x) \cong k[x]/f(x)$ for the minimal polynomial f of x. As K/k is separable, we have

$$k[x] \xrightarrow{d} \Omega_{k[x]/k} \quad , \quad f \longmapsto df \neq 0 \quad \text{non-zero.}$$

Thus: $\Omega_{K/k} \cong \operatorname{coker}\left(f \xrightarrow{d} k[x]/(f(x)) \cdot dx \right) = 0$

because df is a generator of $k[x]/f(x)\, dx$ (as a non-zero vector in a 1-dim. K-v.s.).

(2) $\Omega_{k(x_1,\dots,x_n)/k} = \Omega_{k[x_1,\dots,x_n]_{(0)}/k} \cong \Omega_{k[x_1,\dots,x_n]/k} \otimes_{k[x_1,\dots,x_n]} k[x_1,\dots,x_n]_{(0)} =$

$$= \left(\bigoplus_i k[x_1,\dots,x_n]\, dx_i \right) \otimes_{k[x_1,\dots,x_n]} k(x_1,\dots,x_n) =$$

$$= \bigoplus_i k(x_1,\dots,x_n)\, dx_i$$

(3) Let $k \subseteq L \subseteq K''$ with K''/L algebraic (and thus by assumption wlog. separable) and L/k purely transcendental (as in (2)). Then

$$\underset{(K'')^{\operatorname{trdeg}(L/k)}}{\underset{\|}{\Omega_{L/k} \otimes_L K''}} \longrightarrow \Omega_{K''/k} \longrightarrow \underset{0}{\underset{\|}{\Omega_{K''/L}}} \longrightarrow 0$$

We claim that in fact

$$\Omega_{K''/k} \cong \Omega_{L/k} \otimes_L K'' = (K'')^{\operatorname{trdeg}(K''/k)}$$

This follows either from the discussion in [Matsumura, §26] or from

Proposition: [cf. Matsumura, Thm. 25.1.]

If $R \longrightarrow A \longrightarrow B$ morphism of R-algebras and B is smooth over A, then

$$\Omega_{A/R} \otimes_A B \hookrightarrow \Omega_{B/R} \quad \text{is injective}$$

But any separable extension is smooth.

(4): $K''' \cong k[x]/(x^p - \alpha)$. Then

$$\Omega_{K'''/k} = \operatorname{coker}\left((x^p - \alpha) \xrightarrow{d} K''' \cdot dx \right) = K''' dx / d(x^p - \alpha) =$$

$$= K''' \cdot dx / p x^{p-1} dx = K''' \cdot dx$$

Ex 3:

Correction: It should be $\mathcal{H}om_{\mathcal{O}_S}(\mathcal{E}_\alpha, \mathcal{E}_\alpha \otimes_{\mathcal{O}_{U_\alpha}} \Omega_{U_\alpha/S})$ instead of $\mathcal{H}om_{\mathcal{O}_{U_\alpha}}(\mathcal{E}_\alpha, \mathcal{E}_\alpha \otimes_{\mathcal{O}_{U_\alpha}} \Omega_{U_\alpha/S})$, as connections are only $\mathcal{O}_S$-linear!

Let $\nabla(U_\alpha)$ be the set of all connections $\nabla: \mathcal{E}_\alpha \to \mathcal{E}_\alpha \otimes_{\mathcal{O}_{U_\alpha}} \Omega_{U_\alpha/S}$ over U_α

Let $U_\alpha \subseteq X$ s.th. $\mathcal{E}_\alpha = \mathcal{E}|_{U_\alpha}$ admits a trivialization $\mathcal{E}_\alpha \cong \mathcal{O}_{U_\alpha}^{r}$ (for some $r \geq 0$). Using the canonical $d \in \mathcal{H}om_{\mathcal{O}_S}(\mathcal{O}_{U_\alpha}, \Omega_{U_\alpha/S})$ define:

$$\nabla_\alpha : \mathcal{E}_\alpha \cong \mathcal{O}_{U_\alpha}^r \longrightarrow (\Omega_{U_\alpha/S})^r \cong \mathcal{O}_{U_\alpha}^r \otimes_{\mathcal{O}_{U_\alpha}} \Omega_{U_\alpha/S} \cong \mathcal{E}_\alpha \otimes_{\mathcal{O}_{U_\alpha}} \Omega_{U_\alpha/S}$$

$$(f_1, \dots, f_r) \longmapsto (df_1, \dots, df_r)$$

(which depends on the chosen trivialization $\mathcal{E}_\alpha \cong \mathcal{O}_{U_\alpha}^r$!).

Claim: ∇_α is a connection

Proof: Let $(f_1, \dots, f_r) \in \mathcal{O}_{U_\alpha}^r(U) \cong \mathcal{E}_\alpha(U)$ and $\lambda \in \mathcal{O}_{U_\alpha}(U)$ (for any $U \subseteq U_\alpha$). Then

$$\nabla_\alpha(\lambda \cdot (f_1, \dots, f_r)) = (d(\lambda f_1), \dots, d(\lambda f_r)) =$$
$$= (\lambda\, df_1 + f_1\, d\lambda, \dots, \lambda\, df_r + f_r\, d\lambda) =$$
$$= \lambda \cdot (df_1, \dots, df_r) + (f_1, \dots, f_r) \otimes d\lambda =$$
$$= \lambda \cdot \nabla_\alpha(f_1, \dots, f_r) + (f_1, \dots, f_r) \otimes d\lambda.$$

$\Rightarrow \nabla(U_\alpha) \neq \emptyset$.

We define an $\mathcal{O}_X(U_\alpha)$-module structure on $\mathcal{H}om_{\mathcal{O}_S}(\mathcal{E}_\alpha, \mathcal{E}_\alpha \otimes_{\mathcal{O}_{U_\alpha}} \Omega_{U_\alpha/S})$ as follows: $\forall s \in \mathcal{O}_X(U_\alpha)$ and $\forall \varphi \in \mathcal{H}om_{\mathcal{O}_S}(\mathcal{E}_\alpha, \mathcal{E}_\alpha \otimes_{\mathcal{O}_{U_\alpha}} \Omega_{U_\alpha/S})$:

$(s \cdot \varphi)(f) := s \cdot (\varphi(f))$ using that $\mathcal{E}_\alpha \otimes_{\mathcal{O}_{U_\alpha}} \Omega_{U_\alpha/S}$ is an $\mathcal{O}_{U_\alpha}$-module.

<u>Warning</u>: This $\mathcal{O}_X(U_\alpha)$-module structure does <u>not</u> coincide with the one defined by

$$(s \cdot \varphi)(f) := \varphi(s \cdot f)$$

as φ is only $\mathcal{O}_S$-linear. Nevertheless the following statement stays true even for this module structure.

Claim:

Let $\nabla \in \nabla(U_\alpha)$ be any element. Then

$$\{\nabla' - \nabla \in Hom_{O_S}(\mathcal{E}_\alpha, \mathcal{E}_\alpha \otimes_{O_{U_\alpha}} \Omega_{U_\alpha/S}) \mid \nabla' \in \nabla(U_\alpha)\} =: \tilde{\nabla}$$

is an $O_X(U_\alpha)$-submodule of $Hom_{O_S}(\mathcal{E}_\alpha, \mathcal{E}_\alpha \otimes_{O_{U_\alpha}} \Omega_{U_\alpha/S})$.

Proof:

It is easy to check that $\tilde{\nabla}$ consists of all $\tilde{\nabla} \in Hom_{O_S}(\mathcal{E}_\alpha, \mathcal{E}_\alpha \otimes \Omega_{U_\alpha/S})$ s.th. $\tilde{\nabla}(\lambda \cdot f) = \lambda \cdot \tilde{\nabla}(f) \quad \forall \lambda \in O_{U_\alpha}, f \in \mathcal{E}_\alpha$.

But this condition is obviously invariant under the $O_X(U_\alpha)$-action defined above.

Ex 4:

By definition of $\Omega_{X/S}$ (by glueing affine parts) we may assume wlog X, S affine. Then this follows from [Matsumura, p. 192], where $\Omega_{X/S}$ is defined as I/I^2 (for $I = \ker(m : O_X(X) \otimes O_X(X) \to O_X(X))$) and the universal property is then checked by hand.