2) (1) As $\mathcal{H}_Z^0(\mathcal{F})(V) \subseteq \mathcal{F}(V)$ $\forall V$, (as ab. groups). we have only to check

i) $\varrho_{UV}: \mathcal{H}_Z^0(\mathcal{F})(U) \longrightarrow \mathcal{H}_Z^0(\mathcal{F})(V)$ welldef.

ii) If $\mathcal{U} = \{V_i\}$ open cover $s_i \in \mathcal{H}_Z^0(\mathcal{F})(V_i)$ s.th. $s_i|_{V_i \cap V_j} = s_j|_{V_i \cap V_j}$.

Then there is one $s \in \mathcal{F}(U)$ with $s|_{V_i} = s_i$. Then

$$\operatorname{supp}(s) = \bigcup_i \operatorname{supp}(s|_{V_i}) = \bigcup_i \operatorname{supp}(s_i) \subseteq \bigcup_i (V_i \cap Z) = U \cap Z.$$

$\Rightarrow s \in \mathcal{H}_Z^0(\mathcal{F})$.

(2). Let $V \subseteq X$ open. Then $U \cap V \subseteq X$ is open and

$$j_*(\mathcal{F}|_U)(V) = \mathcal{F}|_U(j^{-1}(V)) = \mathcal{F}|_U(V \cap U) = \varinjlim_{W \supseteq V \cap U} \mathcal{F}(W) = \mathcal{F}(V \cap U).$$

Then define

$$\alpha: \mathcal{F} \longrightarrow j_*(\mathcal{F}|_U) \text{ via } \mathcal{F}(V) \xrightarrow{\varrho_{V, V\cap U}} \mathcal{F}(V \cap U) = j_*(\mathcal{F}|_U)(V).$$

(+ check that α is compatible with restrictions).

Consider now

$$0 \to \mathcal{H}_Z^0(\mathcal{F})(V) \longrightarrow \mathcal{F}(V) \xrightarrow{\alpha} j_*(\mathcal{F}|_U)(V) = \mathcal{F}(V \cap U)$$

Then it is obvious that

$$\ker(\alpha) = \{ s \in \mathcal{F}(V) \mid s|_{V \cap U} = 0\} = \{ s \in \mathcal{F}(V) \mid \operatorname{supp}(s) \subseteq Z \cap U\} = \mathcal{H}_Z^0(\mathcal{F})(V).$$

Remark: $\mathcal{F} \longrightarrow j_*(\mathcal{F}|_U)$ is in general not surjective!

Example: $X = \operatorname{Spec} k[x]$, $Z = V(x)$, $U = \operatorname{Spec} k[x][\frac{1}{x}] = D_x$

Then let $x \in D_f \subseteq X$ and we get

$$j_*(\mathcal{O}_X|_U)(D_f) = \mathcal{O}_X(D_f \cap D_x) = \mathcal{O}_X(D_{f\cdot x}) = k[x][\tfrac{1}{x}, \tfrac{1}{f}] = $$
$$= \mathcal{O}_X(D_f)[\tfrac{1}{x}]$$

$$\Rightarrow (j_*(\mathcal{O}_X|_U))_{\{(x)\}} = \varinjlim_{D_f} \mathcal{O}_X(D_f)[\tfrac{1}{x}] = \Big(\varinjlim_{D_f} \mathcal{O}_X(D_f)\Big)[\tfrac{1}{x}] = k[x]_{(x)}[\tfrac{1}{x}] =$$
$$= k(x).$$

$$\Rightarrow \quad \mathcal{O}_{X,\{(x)\}} \longrightarrow (j_*(\mathcal{O}_X|_U))_{\{(x)\}}$$
$$\begin{matrix} \| & & \| \\ k[x]_{(x)} & \longrightarrow & k(x) \end{matrix} \quad \text{not surjective.}$$

3) If $g \in (f)$ i.e. $g = df$ define $R_f \to R_g$, $\frac{a}{f^n} \longmapsto \frac{ad^n}{g^n}$. This is a directed system.

We show that $\varinjlim R_f$ has the universal property of R_p. Take for this $R \xrightarrow{h} S$ with $h(R \setminus p) \subseteq S^\times$. (note that $R \to \varinjlim R_f$ has this property.) We wish to find the unique factorization over $\varinjlim R_f$:

As $\forall f \in R \setminus p$ $h(f) \in S^\times$ there exist a unique factorization of R_f.

By uniqueness we get a comm. diagr.

$$\begin{tikzcd}
 & R_g \arrow[r] \arrow[rr, bend left] & \varinjlim R_f \arrow[r, dashed, "g"] & S \\
R \arrow[ur, dashed, "h"] \arrow[dr] & & & \\
 & R_f \arrow[uu] \arrow[uur] \arrow[uurr, bend right] & &
\end{tikzcd}$$

Thus we find a unique $g : \varinjlim R_f \to S$ s.th.

$$\begin{tikzcd}
R \arrow[r, "h"] \arrow[d] & S \\
\varinjlim R_f \arrow[ur, dashed, "g"'] &
\end{tikzcd}
\implies \varinjlim R_f \text{ has univ. property of } R_p.$$

4) Claim 1: $\mathcal{F}(U) = \{ s : U \to \coprod_{p \in U} R_p \mid \ldots \}$ is a sheaf.

Proof: obvious.

Let $\mathcal{B} = \{ D_f \mid f \in R \}$ be a basis of $X = \operatorname{Spec} R$. Then define

$$\vartheta : \mathcal{O}_X \longrightarrow \mathcal{F}$$

on $\mathcal{B}$ via:
$$\vartheta_{D_f} : \mathcal{O}_X(D_f) = R_f \longrightarrow \mathcal{F}(D_f) = \{ s : D_f \to \coprod_{f \notin p} R_p \}$$
$$a \longmapsto \vartheta_{D_f}(a)(p) = a \in R_p.$$

Claim 2: ϑ extends uniquely to a morph. of sheaves.

Proof: Indeed $\forall\, U \subseteq X$ open set
$$\vartheta : \mathcal{O}_X(U) = \varprojlim_{V \subseteq U_i, V \in \mathcal{B}} \mathcal{O}_X(V) \xrightarrow{\varprojlim \vartheta_V} \varprojlim_{V \subseteq U_i, V \in \mathcal{B}} \mathcal{F}(V) = \mathcal{F}(U)$$

Claim 3: ϑ is an isomorphism.

Proof: Let's check this on stalks: let $\overset{p}{\overset{\shortparallel}{x}} \in \operatorname{Spec} X$ point. As $\mathcal{B}$ is a basis we have:
$$\mathcal{O}_{X,x} = \varinjlim_{x \in U} \mathcal{O}_X(U) = \varinjlim_{x \in D_f \in \mathcal{B}} \mathcal{O}_X(D_f) = \varinjlim_{x \in D_f \in \mathcal{B}} R_f = R_p.$$
and obviously
$$\mathcal{F}_x = R_p$$
$$\implies \mathcal{O}_{X,x} \cong \mathcal{F}_x$$