1) (1): Note that it suffices to show the glueing property of sections for coverings $\{U_i\}$ s.th. $\nexists\, i \neq j$ with $U_i \subseteq U_j$.

For $U = \emptyset, \{p\}, X_1, X_2$ every open cover of U contains the element U

$\Longrightarrow$ glueing property trivially true

For $U = X$ only have to check cover $\{X_1, X_2\}$. But

$$\{(s_1, s_2) \in \mathcal{O}_X(X_1) \times \mathcal{O}_X(X_2) \mid s_1 = s_2 \in \mathcal{O}_X(X_1 \cap X_2)\} =$$
$$= \{(s_1, s_2) \in k[x]_{(x)} \times k[x]_{(x)} \mid s_1 = s_2 \in k(x)\} = k[x]_{(x)}$$

(as $k[x]_{(x)} \hookrightarrow k(x)$ injective).

Furthermore $X_1 \cong \operatorname{Spec} k[x]_{(x)} \cong X_2$ as Schemes

$\Longrightarrow$ X is scheme

(2) Assume X affine. Then

$$\underset{\substack{\uparrow \\ 3\text{ pts.}}}{X} \cong \operatorname{Spec} \mathcal{O}_X(X) = \underset{\substack{\uparrow \\ 2\text{ pts}}}{\operatorname{Spec}\, k[x]_{(x)}} \quad \lightning.$$

2)
$$\begin{tikzcd}
\operatorname{Spec} K \arrow[r, "g"] \arrow[d, "i"'] & \cancel{\mathrm{Spec}}\ X \arrow[d] \\
\operatorname{Spec} R \arrow[r] \arrow[ur, dashed, "f_i"] & \operatorname{Spec} k
\end{tikzcd}$$

Let $X = \operatorname{Spec} k[x_1] \cup \operatorname{Spec} k[x_2]$ (identifying $k[x_1][\tfrac{1}{x_1}] \overset{x_1 = x_2}{\cong} k[x_2][\tfrac{1}{x_2}]$)

Let $\eta \in X$ generic point (i.e. zero ideal).

Definition of g:

$$g : |\operatorname{Spec} K| \longrightarrow |X|$$
$$(0) \longmapsto \eta$$

$$g^{\#}: \mathcal{O}_X \longrightarrow g_* \mathcal{O}_{\operatorname{Spec} K}$$

given by $\mathcal{O}_X(U) \xrightarrow{\text{restr.}} \mathcal{O}_{X,\eta} \cong k(x) = (g_* \mathcal{O}_{\operatorname{Spec} K})(U)$

$(\forall\, U \neq \emptyset)$.

Definition of f_i $(i = 1, 2)$

$$f_i : |\operatorname{Spec} R| \longrightarrow |X|$$
$$(0) \longmapsto \eta$$
$$(x) \longmapsto (x_i) \in \operatorname{Spec} k[x_i] \subseteq X.$$

$f_i^\#: \mathcal{O}_X \longrightarrow f_{i*}\mathcal{O}_{\operatorname{Spec} R}$ given by:

- If $U \subseteq X$ with $(x_i) \notin U$:

$$\mathcal{O}_X(U) \xrightarrow{\text{restr.}} \mathcal{O}_{X,\eta} \cong k(x) = \mathcal{O}_{\operatorname{Spec} R}((0)) = (f_{i*}(\mathcal{O}_{\operatorname{Spec} R}))(U)$$

- If $U \subseteq X$ with $(x_i) \in U$

$$\mathcal{O}_X(U) \xrightarrow{\text{restr.}} \mathcal{O}_{X,(x_i)} \cong k[x]_{(x)} = \mathcal{O}_{\operatorname{Spec} R}(\operatorname{Spec} R) = (f_{i*}\mathcal{O}_{\operatorname{Spec} R})(U).$$

It remains to see $f_i \circ \iota = \alpha$ $\forall i = \{1,2\}$:

This is obvious on topol. spaces. On sheaves we have

Case 1: $(x_i) \notin U$

$$\begin{tikzcd}
(f_i \circ \iota)^\#: \mathcal{O}_X(U) \arrow[r, "f_i^\#"] \arrow[dd, equal] & \mathcal{O}_{X,\eta} \arrow[r, phantom, "\cong"] \arrow[dd, equal] & \mathcal{O}_{\operatorname{Spec} R}((0)) \arrow[r, "\iota^\#"] \arrow[d, equal] & \mathcal{O}_{\operatorname{Spec} K}((0)) \arrow[d, equal] \\
 & & k(x) \arrow[r, equal] \arrow[d, equal] & k(x) \arrow[d, equal] \\
\alpha^\#: \mathcal{O}_X(U) \arrow[r, "\alpha^\#"] & \mathcal{O}_{X,\eta} \arrow[r, phantom, "\cong"] & \mathcal{O}_{\operatorname{Spec} K}((0)) \arrow[r, equal] & \mathcal{O}_{\operatorname{Spec} K}((0))
\end{tikzcd}$$

and everything commutes.

Case 2: $(x_i) \in U$

$$\begin{tikzcd}
(f_i \circ \iota)^\#: \mathcal{O}_X(U) \arrow[r, "f_i^\#"] \arrow[ddd, equal] \arrow[dddr, phantom, "\circledast"] & \mathcal{O}_{X,(x_i)} \arrow[r, phantom, "\cong"] & \mathcal{O}_{\operatorname{Spec}(R)}(\operatorname{Spec} R) \arrow[r, "\iota^\#"] \arrow[d, equal] & \mathcal{O}_{\operatorname{Spec} K}(\operatorname{Spec} K) \arrow[d, equal] \\
 & & k[x]_{(x)} \arrow[r, hook] \arrow[d] & k(x) \arrow[d, equal] \\
 & & k(x) \arrow[r, equal] \arrow[d, equal] & k(x) \arrow[d, equal] \\
\alpha^\#: \mathcal{O}_X(U) \arrow[r, "\alpha^\#"] & \mathcal{O}_{X,\eta} \arrow[r, phantom, "\cong"] & \mathcal{O}_{\operatorname{Spec} K}(\operatorname{Spec} K) \arrow[r, equal] & \mathcal{O}_{\operatorname{Spec} K}(\operatorname{Spec} K)
\end{tikzcd}$$

Commutativity of $\circledast$:

Note that $\alpha^\#$ and $f_i^\#$ factor over $\mathcal{O}_X(U \cap \operatorname{Spec} k[x_i])$. Hence assume wlog $U \subseteq \operatorname{Spec} k[x_i]$. Then the commutativity is simply

$$\begin{tikzcd}
\mathcal{O}_{\mathbb{A}^1}(U) \arrow[r, "\text{restr.}"] \arrow[ddd, equal] & \mathcal{O}_{\mathbb{A}^1,(x_i)} \arrow[r, phantom, "\cong"] \arrow[d, equal] & \mathcal{O}_{\operatorname{Spec}(R)}(\operatorname{Spec} R) \arrow[d, equal] \\
 & k[x_i]_{(x_i)} \arrow[r, equal] \arrow[d] & k[x]_{(x)} \arrow[d] \\
 & k(x_i) \arrow[r, equal] \arrow[d, equal] & k(x) \arrow[d, equal] \\
\mathcal{O}_{\mathbb{A}^1}(U) \arrow[r, "\text{restr.}"] & \mathcal{O}_{\mathbb{A}^1,(0)} \arrow[r, equal] & \mathcal{O}_{\operatorname{Spec} K}(\operatorname{Spec} K)
\end{tikzcd}$$

Less explicit but faster:

Both α and $f_i \circ \iota$ factor over $\operatorname{Spec} k[x_i] \subseteq X$. Thus it suffices to show:

$$\begin{tikzcd} \operatorname{Spec} K \arrow[d, "\iota"'] \arrow[dr, "\alpha"] \arrow[dr, phantom, shift right=3ex, "G"] & \\ \operatorname{Spec} R \arrow[r, "f_i"'] & \operatorname{Spec} k[x_i] \subseteq X \end{tikzcd}$$

This can be done by considering the morphisms between global sections (everything is affine!):

$$\begin{tikzcd} K & \\ R \arrow[u, "\iota^\#"] & k[x_i] \arrow[ul, "\alpha^\#"'] \arrow[l, "f_i^\#"] \end{tikzcd}$$

But this is nothing else than

$$\begin{tikzcd} k(x) & \\ k[x]_{(x)} \arrow[u, hook] & k[x_i] \arrow[ul, "x \mapsfrom x_i"'] \arrow[l, "x \mapsfrom x_i"] \end{tikzcd}$$

which commutes.

3+4)

WARNING: The presheaf Nil defined by $Nil(U) = Nil(\mathcal{O}_X(U))$ (nilradical) is <u>not</u> a sheaf (unless X quasi-compact noeth.)

Example: $X = \coprod_{i>0} X_i$, $X_i = \operatorname{Spec} k[x]/(x^i)$. Then consider the open cover $\{X_i\}_{i>0}$ of X. Then

$$x \in Nil(X_i) \subseteq \mathcal{O}_X(X_i) \quad \forall i$$

but $(x, x, x, \dots) \in \mathcal{O}_X(X)$ is <u>not</u> in $Nil(X)$!

(1) If $s_x \in \mathcal{O}_{X,x}$ nilpotent (for some x) $\Rightarrow s_x^n = 0 \in \mathcal{O}_{X,x}$

$\Rightarrow \exists$ representative: $s \in \mathcal{O}_X(U)$ $(x \in U \subseteq X)$ with $s_x^n = 0$

$\Rightarrow (X, \mathcal{O}_X)$ not reduced.

If $s \in \mathcal{O}_X(V)$ nilpotent (for some $V \subseteq X$) $\Rightarrow s^n = 0 \in \mathcal{O}_X(V)$

$\Rightarrow$ ~~then~~ $\forall x \in V$: $s_x^n = 0 \in \mathcal{O}_{X,x} \Rightarrow \forall x \in X$ $\mathcal{O}_{X,x}$ non-reduced.

(2) As $(X, (\mathcal{O}_X)_{red})$ is a ringed space it suffices ~~to show~~ that it is locally an affine scheme. Pick any $U \subseteq X$ open s.th.

$(U, \mathcal{O}_X|_U) \cong \operatorname{Spec} R$ is affine.

Claim: $(U, (\mathcal{O}_X)_{red}|_U) \cong \operatorname{Spec} R_{red} = \operatorname{Spec} R/Nil(R)$.

Proof:

$$0 \to Nil(R) \to R \to R/Nil(R) \to 0$$

$$\rightsquigarrow 0 \to Nil(R)_f \to R_f \to (R/Nil R)_f \to 0 \quad \forall f \in R$$

$$\Rightarrow \mathcal{O}_{\operatorname{Spec} R/Nil(R)}(D_f) = (R/Nil(R))_f \cong R_f/Nil(R)_f = R_f/Nil(R_f)$$
$$= (R_f)_{red} = (\mathcal{O}_X(D_f))_{red}$$

$\Rightarrow$ The sheaf $\mathcal{O}_{\operatorname{Spec} R/Nil(R)}$ and the presheaf defined by $V \mapsto \mathcal{O}_X(V)_{red}$ coincide on a basis of U.

$\Rightarrow$ The sheaves $\mathcal{O}_{\operatorname{Spec} R/Nil(R)}$ and $(\mathcal{O}_X)_{red}|_U$ coincide

$\Rightarrow (U, (\mathcal{O}_X)_{red}|_U) \cong \operatorname{Spec} R/Nil(R)$.

(3) Define $r: X_{red} \to X$ as the identity on topol. spaces and on sheaves as the sheafification of $\mathcal{O}_X(U) \to \mathcal{O}_X(U)_{red}$ $\forall U \subseteq X$. (which induces $r^\#: \mathcal{O}_X(U) \to r_*(\mathcal{O}_{X_{red}})(U) = \mathcal{O}_{X_{red}}(U)$.

(4):

Claim 1: The assertion is true for $X = \operatorname{Spec} R$ affine.

Proof:

Then $X_{red} = \operatorname{Spec} R/Nil(R)$ is affine as well. Then the global sections of f define:

$$\begin{tikzcd} \mathcal{O}_Y(Y) & R \arrow[l, "f^{\#}"'] \arrow[d, "pr."] \\ & R_{red} \arrow[ul, dashed, "g^{\#}"] \end{tikzcd}$$

As Y is reduced, $Nil(R) \subseteq \ker(f^{\#} : R \to \mathcal{O}_Y(Y))$. Hence $f^{\#}: R \to \mathcal{O}_Y(Y)$ uniquely factors over R_{red} and gives $g^{\#}: R_{red} \to \mathcal{O}_Y(Y)$. Via $\operatorname{Hom}(Y, X_{red}) \cong \operatorname{Hom}(R_{red}, \mathcal{O}_Y(Y))$ we get a unique morphism $g: Y \to X_{red}$:

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[dr, dashed, "g"'] & X \\ & X_{red} \arrow[u, hook] \end{tikzcd}$$

Claim 2: The assertion is true for all X.

Proof:

Let $X = \bigcup \{U_i\}$ be an open cover of X by affines U_i. Then we have $\forall i$ a unique factorization

$$\begin{tikzcd} f^{-1}(U_i) \arrow[r, "f_i"] \arrow[dr, "g_i"'] & U_i \\ & U_{i\,red} \arrow[u, hook] \end{tikzcd}$$

By uniqueness of g_i we have $g_i = g_j : f^{-1}(U_i) \cap f^{-1}(U_j) \to (U_i \cap U_j)_{red} \subseteq X_{red}$.

Hence the g_i glue to a unique morphism

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[dr, dashed, "\exists!\, g"'] & X \\ & X_{red} \arrow[u, hook] \end{tikzcd}$$