1) $D_+(x_i) = \operatorname{Spec}\left(R[x_0,\dots,x_n][\frac{1}{x_i}]\right)_0 = \operatorname{Spec} R[\frac{x_0}{x_i},\dots,\frac{x_n}{x_i}] \cong \mathbb{A}^n_R$.

Assume $p \in \mathbb{P}^n_R \setminus \bigcup_i D_+(x_i)$. As $p \notin D_+(x_i)$ we have $x_i \in p \; \forall i$.

$\Rightarrow S_+ = (x_1,\dots,x_n) \subseteq p$ ↯.

$U_{ij} = D_+(x_i) \cap D_+(x_j) = D_+(x_i x_j) = \operatorname{Spec}\left(R[x_0,\dots,x_n][\frac{1}{x_i},\frac{1}{x_j}]\right)_0 =$

$= \operatorname{Spec} R[\frac{x_0}{x_i},\dots,\frac{x_n}{x_i},\frac{x_0}{x_j},\dots,\frac{x_n}{x_j}] = \operatorname{Spec} R[\frac{x_0}{x_i},\dots,\frac{x_n}{x_i}][\frac{x_i}{x_j}]$.

Then U_{ij}'s are affine and it suffices to give φ_{ij} on the corresp. rings:

$$\varphi_{ij}^\#: R[\frac{x_0}{x_j},\dots,\frac{x_n}{x_j}][(\frac{x_i}{x_j})^{-1}] \longrightarrow R[\frac{x_0}{x_i},\dots,\frac{x_n}{x_i}][(\frac{x_j}{x_i})^{-1}]$$

$$\frac{x_k}{x_j} \longmapsto \frac{x_k}{x_i}\cdot\left(\frac{x_j}{x_i}\right)^{-1}.$$

2) Claim 1: Via s, $\mathcal{O}_X$ is a sheaf of ~~$\mathcal{O}_X$~~ $\mathbb{F}_p$-algebras.

Proof: $s|_U: U \to \operatorname{Spec}\mathbb{F}_p$ is given by $\mathbb{F}_p \longrightarrow \mathcal{O}_X(U)$

$\rightsquigarrow \mathcal{O}_X(U)$ has a $\mathbb{F}_p$-alg.

As
$$\begin{tikzcd} U \arrow[r, "s|_U"] & \operatorname{Spec}\mathbb{F}_p \\ V \arrow[u, hook] \arrow[ur, "s|_V"'] & \end{tikzcd}$$
commutes $\rightsquigarrow$
$$\begin{tikzcd} & \mathcal{O}_X(U) \arrow[d] \\ \mathbb{F}_p \arrow[ur] \arrow[r] & \mathcal{O}_X(V) \end{tikzcd}$$
commutes

$\Rightarrow$ restriction morph. on $\mathcal{O}_X$ are $\mathbb{F}_p$-linear!

Claim 2: $F^\#: \mathcal{O}_X \to F_*\mathcal{O}_X$ is morph. of sheaves of rings.

Proof: $x \longmapsto x^p$ defines a morph. of rings on every $\mathbb{F}_p$-algebra.

Let $V \subseteq U \subseteq X$ open. $x \in \mathcal{O}_X(U)$. Then as $\rho_{UV}: \mathcal{O}_X(U) \to \mathcal{O}_X(V)$ is a ring morph.: $F^\#(\rho_{UV}(x)) = \rho_{UV}(x)^p = \rho_{UV}(x^p) = \rho_{UV}(F^\#(x))$.

Claim 3: $(F,F^\#): (X,\mathcal{O}_X) \to (X,\mathcal{O}_X)$ is morph. of schemes.

Proof: It remains to see that $F^\#$ is a local morphism.

Let $x \in X$ with stalk $\mathcal{O}_{X,x}$. Then $F^\#: \mathcal{O}_{X,F(x)} \to \mathcal{O}_{X,x}$ is simply $F^\# = (\cdot)^p: \mathcal{O}_{X,x} \to \mathcal{O}_{X,x}$. Let $m_x \subseteq \mathcal{O}_{X,x}$ be the max. ideal. Then $F^\#(m_x) \subseteq m_x^p \subseteq m_x \Rightarrow F^\#$ local.

3)+4)

(1): I is obviously generated by $\{a \in I \mid a \text{ homog.}\}$..

(2): If $I = (a_i)$ and $J = (b_j)$ homog. generated ideals. Then

$I+J = (a_i, b_j)_{i,j}$, $I\cdot J = (a_i \cdot b_j)_{i,j}$ homog. generated.

Intersection obvious by original def.

Radical:

Assume $\sqrt{I}$ not homog. $\Rightarrow \exists\, f \in \sqrt{I}$, $f = \sum_{i=0}^{N} f_i$, $f_i \in R_i$ s.th. not all $f_i \in \sqrt{I}$. Pick $N_0 \in \mathbb{N}$ s.th. $f_{N_0} \notin \sqrt{I}$ but $f_i \in \sqrt{I}\ \forall i < N_0$

$\Rightarrow \sum_{i=0}^{N_0-1} f_i \in \sqrt{I}$

$\Rightarrow \tilde{f} := f - \sum_{i=0}^{N_0-1} f_i \in \sqrt{I}$

$\Rightarrow \exists n \in \mathbb{N}: \tilde{f}^n \in I$

Consider the degree $= n \cdot N_0$ - part of $\tilde{f}^n$:

$I \ni (\tilde{f}^n)_{n\cdot N_0} = (f_{N_0})^n$ as $\tilde{f}$ has no part of deg $< N_0$!

$\Rightarrow f_{N_0} \in \sqrt{I}$ ↯.

(3) "only if": obvious

"if": Let $f = \sum f_i \notin P$, $g = \sum g_j \notin P$ with $f \cdot g \in P$. Choose $N_1, N_2 \in \mathbb{N}$ s.th. $f_{N_1} \notin P$, $f_i \in P\ \forall i < N_1$, $g_{N_2} \notin P$, $g_j \in P\ \forall j < N_2$.

$\Rightarrow (f - \sum_{i=0}^{N_1} f_i) \cdot (g - \sum_{j=0}^{N_2} g_j) \in P$

Consider degree $= N_1 + N_2$ - part:

$f_{N_1} \cdot g_{N_2} = \left((f - \sum_{i=0}^{N_1} f_i)(g - \sum_{j=0}^{N_2} g_j)\right)_{N_1+N_2} \in P$, $f_{N_1} \cdot g_{N_2} \notin P$

Contradiction to assumption!

(4) (a) $\forall\, I \subseteq P$ prime: $R_+ \subseteq P$

(b) $\sqrt{I} = R$ or $\sqrt{I} = R_+$

(b') $R_+ \subseteq \sqrt{I}$

(c) $R_d \subseteq I$ for some $d \geq 1$.

Proof of (a) $\Rightarrow$ (b')

Let $I \subseteq R$ with (a). Then by def. $\mathrm{Proj}(R/I) = \emptyset$. Then let $f \in (R/I)_+$ and consider $\mathrm{Spec}\left((R/I)[\tfrac{1}{f}]\right)_0 \cong D_+(f) \subseteq \mathrm{Proj}(R/I) = \emptyset$

$\Rightarrow \mathrm{Spec}((R/I[\tfrac{1}{f}])_0) = \emptyset \Rightarrow (R/I[\tfrac{1}{f}])_0 = 0$.

$\Rightarrow 1 = 0$ in $R/I[\tfrac{1}{f}] \Rightarrow R/I[\tfrac{1}{f}] = 0$

$\Rightarrow f$ nilpotent in $R/I \Rightarrow f \in \sqrt{I}$

$\Longrightarrow R_+ \subseteq \sqrt{I}$.

Proof of (b') $\Rightarrow$ (a): obvious

Proof of (b) $\Leftrightarrow$ (b')

$$\{I \subseteq R \text{ homog. with } R_+ \subseteq I\} \cong \{I \subseteq R/R_+ \text{ homog.}\}$$
$$\cong \{I \subseteq R_0\} = \{(0) \subseteq R_0 = k \text{ and } k \subseteq R_0 = k\}$$

But $(0) \subseteq R_0$ corresponds to $I = R_+$ and $k \in R_0$ corresponds to $I = R$.

Proof of (b') $\Rightarrow$ (c)

Let $I \subseteq R$ homog. with $R_+ \subseteq \sqrt{I}$. Then R/I is a fin. gen. k-alg. ~~with~~, i.e. noeth. Thus $(R/I)_+$ is fin. generated by (wlog. homog.) elements $f_1, \dots, f_n$.
As $R_+ \subseteq \sqrt{I}$ each f_i is nilpotent, i.e. $f_i^{n_i} = 0$ for some $n_i \in \mathbb{N}$. Let $d_i = \deg f_i$.

$\cancel{\Rightarrow ((R/I)_+)^{\sum n_i} = (f_1, \dots, f_n)^{\sum n_i}}$

Claim: $(R/I)_N = 0$ for $N > \sum_i n_i d_i$.

Proof: As the $f_1, \dots, f_n$ generate R/I as k-algebra, any $g \in (R/I)_N$ may be written as $\sum_j (\lambda_j \prod_i f_i^{\alpha_{ij}})$ for $\lambda_j \in k$, $\alpha_{ij} \in \mathbb{N}$ s.th. $\sum \alpha_{ij} d_i = N$.
But by choice of N $\forall j\ \exists i$ with $\alpha_{ij} \geq n_i \Rightarrow$ Each summand is $= 0$!

$\Rightarrow (R/I)_N = 0 \Rightarrow R_N \subseteq I$.

Proof of (c) $\Rightarrow$ (b'):

Let $R_d \subseteq I$ for some d. But $R_+{}^d \subseteq R_d$ by def. Hence $R_+ \subseteq \sqrt{I}$ as desired.