1) Let $\operatorname{Spec} R = Z_1 \amalg Z_2$ (with Z_1, Z_2 nonempty). Then $Z_i = V(I_i)$ $(i=1,2)$ for some ideals $I_i \subseteq R$. Disjoint union $\Rightarrow$ $I_1 + I_2 = R$, $I_1 \cap I_2 \subseteq Nil(R)$.

$\Rightarrow \exists\, \tilde{e}_i \in I_i$ with $\tilde{e}_1 + \tilde{e}_2 = 1$ and $\tilde{e}_1\tilde{e}_2 \in Nil(R)$ (and $(\tilde{e}_1\tilde{e}_2)^n = 0$).

$\Rightarrow 1 - \tilde{e}_1^n - \tilde{e}_2^n = (\tilde{e}_1 + \tilde{e}_2)^n - \tilde{e}_1 - \tilde{e}_2 \in (\tilde{e}_1\tilde{e}_2)R \subseteq Nil(R)$

$\Rightarrow \tilde{e}_1^n + \tilde{e}_2^n \in R^\times$

$\Rightarrow \exists\, a \in R^\times$ with $e_i = a \cdot \tilde{e}_i^n$ satisfying $e_1 + e_2 = 1$

Furthermore by def. $e_1 e_2 = 0 \Rightarrow e_1^2 = e_1^2 + e_1e_2 = e_1(e_1+e_2) = e_1$ (same for e_2)

Finally $e_1 = 0 \Rightarrow e_2 = 1 \Rightarrow \tilde{e}_2 \in R^\times \Rightarrow I_2 = R \Rightarrow Z_2 = \emptyset$ ↯

Let $e_1, e_2 \in R$ orth. idempotents. By Chinese remainder thm:

$R \cong R/(e_1) \times R/(e_2)$ as $(e_1) + (e_2) = R$ (obvious) and

$(e_1) \cap (e_2) = 0$: Let $\lambda = ae_1 = be_2 \in (e_1)\cap(e_2)$, $a, b \in R$. (*)

$\Rightarrow \lambda = be_2 = b(e_1+e_2)e_2 = (be_2 + ae_1)e_2 = (a+b)e_1e_2 = 0$ $\square$

Check: $R/(e_i) \neq 0$: Obvious as $e_1 \neq 0 \in R/(e_2)$ and vice versa. (cf. (*))

Let $R \cong R_1 \times R_2$. Then $\operatorname{Spec} R = V(0,1) \amalg V(1,0) = \operatorname{Spec} R_1 \amalg \operatorname{Spec} R_2$.

2) (1): Let $Y \subseteq X$, Y irred. Let $\overline{Y} = \bigcup Z_i$, $Z_i \subseteq \overline{Y}$ closed.

$\Rightarrow Y = \bigcup (Z_i \cap Y)$ and by irred. $\exists j$: $Y \subseteq Z_j \cap Y$.

$\Rightarrow \overline{Y} \subseteq \overline{Z_j} = Z_j$. $\Rightarrow \overline{Y}$ irred.

(2) "$\Rightarrow$" Wlog $I = \sqrt{I}$. ~~Let~~ Assume $a, b \in R \setminus I$ with $a \cdot b \in I$.

$\Rightarrow V(I) \subseteq V(a) \cup V(b)$, $V(I) \not\subseteq V(a)$, $V(I) \not\subseteq V(b)$.

$\Rightarrow V(I) \subseteq (V(a) \cap V(I)) \cup (V(b) \cap V(I))$, " "

$\Rightarrow V(I)$ not irred.

"$\Leftarrow$" Let ~~Assume~~ $V(I) = V(\sqrt{I})$ with $\sqrt{I}$ prime.

Let $V(\sqrt{I}) = V(J_1) \cup V(J_2) = V(J_1 \cdot J_2)$.

$\Rightarrow J_1 \cdot J_2 \subseteq \sqrt{I}$

$\Rightarrow$ As $\sqrt{I}$ prime: $J_1 \subseteq \sqrt{I}$ or $J_2 \subseteq \sqrt{I}$.

$\Rightarrow V(\sqrt{I}) \subseteq V(J_1)$ or $V(\sqrt{I}) \subseteq V(J_2)$

$\Rightarrow V(\sqrt{I})$ irred.

(3) obvious.

3) (1) Let $s \in S$ closed. Then $\kappa(s) = k$ (as it is a finite extension of k!!).

Furthermore
$$X \times_S s \cong \operatorname{Spec}\left(k[x,y] \otimes_{k[x]} k[x]/m_s\right) \cong$$
$$\cong \operatorname{Spec}\left(k[x]/m_s\,[y]\right) \cong \operatorname{Spec}(\kappa(s)[y]) \cong$$
$$\cong \operatorname{Spec}(k[y]) \cong \mathbb{A}^1_k$$

(with $m_s \subseteq k[x]$ the max. ideal corresp. to the closed pt. s)

Let $s \in S$ not closed. Then $s = \eta$ generic pt. corresponding to $(0) \subseteq k[x]$

$\Rightarrow \kappa(s) = k[x]_{(0)} \cong \operatorname{Frac}(k[x]) = k(x)$.

$\Rightarrow X \times_S \eta \cong \operatorname{Spec}\left(k[x,y] \otimes_{k[x]} k(x)\right) \cong \operatorname{Spec}(k(x)[y]) = \mathbb{A}^1_{k(x)}$.

(2) If $s = (p) \in \operatorname{Spec}\mathbb{Z}$ for p prime. Then as above $\kappa(s) = \mathbb{F}_p$ and
$X \times_{\operatorname{Spec}\mathbb{Z}} s \cong \mathbb{A}^1_{\mathbb{F}_p}$.

If $s = (0) \in \operatorname{Spec}\mathbb{Z}$ generic, then $\kappa(s) = \mathbb{Q}$ and
$X \times_{\operatorname{Spec}\mathbb{Z}} \eta \cong \mathbb{A}^1_{\mathbb{Q}}$.

4) Claim: Let L/k any field extension. Then there is a bijection:
$$\left\{ \begin{matrix} \operatorname{Spec} L[\varepsilon]/(\varepsilon^2) \longrightarrow X \\ \text{with image } x \in X \end{matrix} \right\} \overset{1:1}{\longleftrightarrow} \left\{ \begin{matrix} k\text{-linear field inclusion } \kappa(x) \hookrightarrow L \text{ and} \\ \text{an element in } \operatorname{Hom}_{\kappa(x)}(m_x/m_x^2, L) \end{matrix} \right\}$$

Lemma 1:
$$\left\{ \begin{matrix} \operatorname{Spec} L[\varepsilon]/(\varepsilon^2) \longrightarrow X \\ \text{with image } x \in X \end{matrix} \right\} \overset{1:1}{\longleftrightarrow} \left\{ \begin{matrix} k\text{-linear local morphisms} \\ \mathcal{O}_{X,x} \longrightarrow L[\varepsilon]/(\varepsilon^2) \end{matrix} \right\}$$

Proof:
$$f: \operatorname{Spec} L[\varepsilon]/(\varepsilon^2) \to X \longmapsto f^\#_x: \mathcal{O}_{X,x} = \mathcal{O}_{X,f((\varepsilon))} \longrightarrow \left(L[\varepsilon]/(\varepsilon^2)\right)_{(\varepsilon)} \cong L[\varepsilon]/(\varepsilon^2).$$

$$\operatorname{Spec} L[\varepsilon]/(\varepsilon^2) \xrightarrow{\alpha} \operatorname{Spec} \mathcal{O}_{X,x} \overset{\text{kanonisch!}}{\hookrightarrow} X \longleftarrow \alpha^\#: \mathcal{O}_{X,x} \longrightarrow L[\varepsilon]/(\varepsilon^2)$$

Lemma 2:
$$\left\{ \begin{matrix} k\text{-linear local morphisms} \\ \mathcal{O}_{X,x}/m_x^2 \longrightarrow L[\varepsilon]/(\varepsilon^2) \end{matrix} \right\} \overset{1:1}{\longleftrightarrow} \left\{ \begin{matrix} k\text{-lin. local morph.} \\ \mathcal{O}_{X,x} \longrightarrow L[\varepsilon]/(\varepsilon^2) \end{matrix} \right\}$$

Proof:

For every $\alpha: \mathcal{O}_{X,x} \to L[\varepsilon]/(\varepsilon^2)$ local: $\alpha(m_x) \subseteq (\varepsilon) \Rightarrow m_x^2 \subseteq \ker(\alpha)$.

Lemma 3:

There is a canonical iso of $\kappa(x)$-vector spaces
$$\mathcal{O}_{X,x}/m_x^2 \cong \mathcal{O}_{X,x}/m_x \oplus m_x/m_x^2 .$$

Proof:

We have a s.e.s.: $0 \to m_x/m_x^2 \to \mathcal{O}_{X,x}/m_x^2 \to \mathcal{O}_{X,x}/m_x \to 0$.

This has a splitting as $\kappa(x)$-vector spaces given by:
$$\mathcal{O}_{X,x}/m_x \longrightarrow \mathcal{O}_{X,x}/m_x^2 , \quad 1 \longmapsto 1.$$

Lemma 4:
$$\mathrm{Hom}_k(\mathcal{O}_{X,x}/m_x^2, L[\varepsilon]/(\varepsilon^2)) \overset{\text{bijection}}{\cong} \mathrm{Hom}_k(\kappa(x), L) \times \mathrm{Hom}_{\kappa(x)}(m_x/m_x^2, L)$$

Proof:

The bijection is given by
$$\begin{array}{ccccc}
\mathcal{O}_{X,x}/m_x^2 & \cong & \mathcal{O}_{X,x}/m_x & \oplus & m_x/m_x^2 \\
\alpha \downarrow & & \beta \downarrow & & \gamma \downarrow \\
L[\varepsilon]/(\varepsilon^2) & \cong & L & \oplus & \varepsilon \cdot L[\varepsilon]/(\varepsilon^2)
\end{array}$$

1) For given $\mathcal{O}_{X,x}/m_x^2 \to L[\varepsilon]/(\varepsilon^2)$ it is obvious that $\mathcal{O}_{X,x}/m_x \to L$ is a morphism between fields (and thus injective) and that $m_x/m_x^2 \to \varepsilon L[\varepsilon]/(\varepsilon^2)$ is $\kappa(x) = \mathcal{O}_{X,x}/m_x$-linear (where it acts on the target via $\kappa(x) \to L$ just constructed).

2) It remains to see, that given $\kappa(x) \to L$ and $m_x/m_x^2 \to \varepsilon L[\varepsilon]/(\varepsilon^2)$, the induced morph. $\mathcal{O}_{X,x}/m_x^2 \to L[\varepsilon]/(\varepsilon^2)$ is multiplicative (and not only k-linear):

Let $a = b + c \in \mathcal{O}_{X,x}/m_x^2 = \mathcal{O}_{X,x}/m_x \oplus m_x/m_x^2$ and similarly $a' = b' + c'$.

Then $a \cdot a' = b \cdot b' + (bc' + b'c)$ is the corresponding decomp. for $a \cdot a'$.

$$\begin{aligned}
\Rightarrow \alpha(a \cdot a') &= \beta(b \cdot b') + \gamma(bc' + b'c) = \\
&= \beta(b) \cdot \beta(b') + \gamma(bc') + \gamma(b'c) \overset{\gamma \text{ is } \kappa(x)\text{-linear!}}{=} \\
&= \beta(b) \cdot \beta(b') + \beta(b)\gamma(c') + \beta(b')\gamma(c) \overset{\gamma(c)\cdot\gamma(c') = 0!}{=} \\
&= (\beta(b) + \gamma(c)) \cdot (\beta(b') + \gamma(c')) = \\
&= \alpha(a) \cdot \alpha(a').
\end{aligned}$$
$\square$

Remark:

The original problem is a special case of our claim:

Let $L = k$. Then $\mathcal{O}_{V,x}$ is a k-algebra and so is $\kappa(x)$. But we have a field inclusion $\kappa(x) \hookrightarrow L = k \Rightarrow \kappa(x) \cong k$.

Hence we get an identification:

$$\underbrace{\mathrm{Hom}_{k(x)}(m_x/m_x^2, L)}_{\text{cf. claim}} = \mathrm{Hom}_{k(x)}(m_x/m_x^2, \kappa(x)) = T_x.$$

$\Rightarrow$ Claim rewrites as the assertion of ex. 4!