1) Let $\mathcal{U} = \operatorname{Spec} R \hookrightarrow X$ open affine. Then

$$\mathcal{O}_{X,\eta} \cong \mathcal{O}_{\mathcal{U},\eta} \cong \mathcal{O}_{\operatorname{Spec} R,(0)} = R_{(0)} \cong \operatorname{Quot}(R).$$

2) (1) Let $\iota: Z \hookrightarrow X$ be a closed immersion.

Claim: $\forall \mathcal{U} \subseteq X$ open : $\iota|_{\iota^{-1}(\mathcal{U})} : \iota^{-1}(\mathcal{U}) \longrightarrow \mathcal{U}$ is again a closed immersion.

Proof:

$\iota|_{\iota^{-1}(\mathcal{U})}$ is obviously a homeo onto a closed subset of $\mathcal{U}$. Furthermore $\forall V \subseteq \mathcal{U}$ open:

$$\begin{array}{ccc}
(\iota|_{\iota^{-1}(\mathcal{U})})^{\#} : \mathcal{O}_{\mathcal{U}}(V) & \longrightarrow & ((\iota|_{\iota^{-1}(\mathcal{U})})_* \mathcal{O}_{\iota^{-1}(\mathcal{U})})(V) = \mathcal{O}_{\iota^{-1}(\mathcal{U})}((\iota|_{\iota^{-1}(\mathcal{U})})^{-1}(V)) \\
 & & = \mathcal{O}_{\iota^{-1}(\mathcal{U})}(\iota^{-1}(V)) \\
\| & & \| \\
\iota^{\#} : \mathcal{O}_X(V) & \longrightarrow & \mathcal{O}_Z(\iota^{-1}(V))
\end{array}$$

is commutative (by def. of the restriction of a morph. to an open subset)

$\Longrightarrow (\iota|_{\iota^{-1}(\mathcal{U})})^{\#} = \iota^{\#}|_{\mathcal{U}}$ is surjective.

Cover now X by open affines $\{\mathcal{U}_i\}_i$. Then $\iota^{-1}(\mathcal{U}_i)$ is again affine (as it is isomorphic to a closed subscheme of $\mathcal{U}_i$) and by the claim we have:

$$\begin{array}{ccc}
\iota^{-1}(\mathcal{U}_i) & \longrightarrow & \mathcal{U}_i \\
\wr\| & & \wr\| \\
\operatorname{Spec} R/I & \longrightarrow & \operatorname{Spec} R
\end{array}$$

for some ring R + ideal $I \subseteq R$.

But R/I is a fin. gen. R-alg!

(2) Let $\{W_i \cong \operatorname{Spec} R_i\}_i$ open affine cover of Z.

Let $\forall i$: $\{V_{ij} \cong \operatorname{Spec} S_{ij}\}_j$ open affine cover of $g^{-1}(W_i)$ (in Y)

Let $\forall i,j$: $\{U_{ijh} \cong \operatorname{Spec} T_{ijh}\}_h$ open affine cover of $f^{-1}(V_{ij})$ (in X)

Then T_{ijh} fin. gen. S_{ij}-alg. (as f of fin. type)

S_{ij} fin. gen. R_i-alg. (as g of fin. type)

$\Longrightarrow T_{ijh}$ fin. gen. R_i-alg

$\Longrightarrow g \circ f$ of finite type.

(3) Let $\{U_i\}_i$ open affine cover of Y

Let $\{V_{ij}\}_j$ " " " of $f^{-1}(U_i) \subseteq X$

Let $\{U'_{ik}\}_k$ " " " of $g^{-1}(U_i) \subseteq Y'$ (where $g: Y' \to Y$).

Then $\{U'_{ik} \times_{U_i} V_{ij}\}_{j}$ defines by construction of $X \times_Y Y'$ an affine open cover of $f'^{-1}(U'_{ik}) \subseteq X \times_Y Y'$ (to construct $X \times_Y Y'$ you glue these $U'_{ik} \times_{U_i} V_{ij}$ together!).

If $U_i \cong \mathrm{Spec}\, R_i$, $V_{ij} \cong \mathrm{Spec}\, S_{ij}$, $U'_{ik} \cong \mathrm{Spec}\, R'_{ik}$, then

$$U'_{ik} \times_{U_i} V_{ij} \cong \mathrm{Spec}(R'_{ik} \otimes_{R_i} S_{ij}) \longrightarrow U'_{ik} = \mathrm{Spec}\, R'_{ik}$$

is given by

$$R'_{ik} \longrightarrow R'_{ik} \otimes_{R_i} S_{ij} \qquad a \longmapsto a \otimes 1.$$

But S_{ij} is a fin. gen. R_i-algebra (as f is of fin. type)

$\Rightarrow R'_{ik} \otimes_{R_i} S_{ij}$ is a fin. gen. R'_{ik}-algebra.

$\Longrightarrow f': X \times_Y Y' \longrightarrow Y'$ is of finite type.

(4) Factor
$$\begin{tikzcd} X \times_Z Y \arrow[rr] \arrow[dr, "g'"'] & & Z \\ & X \arrow[ur, "f"'] & \end{tikzcd}$$
Then g' is of fin. type by (3) and $f \circ g'$ is of fin. type by (2) $\square$

4) "The Beginner's Proof"

Claim 1:

Let $f_X : \tilde{X} = \mathrm{Spec}\, \tilde{R} \longrightarrow X = \mathrm{Spec}\, R$, $U = D(f) \subseteq X$ standard affine open.

Then $f_X^{-1}(U) \cong \tilde{U}$

Proof:

$$f_X^{-1}(U) = \tilde{X} \times_X U = \mathrm{Spec}\, \tilde{R} \otimes_R R_f \cong \mathrm{Spec}\, (\tilde{R})_f = \mathrm{Spec}\, \widetilde{R_f} = \tilde{U}.$$

Claim 2:

Let X be any scheme, $U, V \subseteq X$ affine open. Then $U \cap V$ can be covered by open affine subsets $\{W_i\}_i$ s.th. W_i is standard open in both U and V, i.e. $\exists\, f_i \in \mathcal{O}_U(U)$, $g_i \in \mathcal{O}_V(V)$ s.th.

$$W_i \cong \cancel{\mathrm{Spec}}\, D(f_i) \subseteq U = \mathrm{Spec}(\mathcal{O}_U(U))$$

$$W_i \cong D(g_i) \subseteq V = \mathrm{Spec}(\mathcal{O}_V(V)).$$

Proof:

Let $\mathcal{U} = \operatorname{Spec} R$, $V = \operatorname{Spec} S$. Let $\{\mathcal{U}_i \cong \operatorname{Spec} R_i\}_i$ be a cover of $\mathcal{U} \cap V$ s.th. $\mathcal{U}_i$ is standard open in $\mathcal{U}$ (standard opens form a basis for the topol!). These $\mathcal{U}_i$ need <u>not</u> to be standard open in V!
But we may cover $\mathcal{U}_i \subseteq \mathcal{U} \cap V \subseteq V$ with affine opens $\{W_{ij}\}_j$ s.th. W_{ij} is standard open in V.

$\Rightarrow \exists\, g_{ij} \in S$ s.th. $W_{ij} \cong D(g_{ij}) = \operatorname{Spec} S[g_{ij}^{-1}] \subseteq V$.

Via $S = \mathcal{O}_V(V) \to \mathcal{O}_V(\mathcal{U}_i) = R_i$, consider the images $\overline{g_{ij}} \in R_i$ of $g_{ij} \in S$. Then $W_{ij} \cong D(\overline{g_{ij}}) = \operatorname{Spec} R_i[\overline{g_{ij}}^{-1}] \subseteq \mathcal{U}_i$ is standard open (the first isom. holds as $W_{ij} \subseteq \mathcal{U}_i$ is exactly the locus, where the function $\overline{g_{ij}}$ does not vanish).

$\Rightarrow W_{ij}$ standard open in $\mathcal{U}_i$ standard open in $\mathcal{U}$

$\Rightarrow W_{ij}$ standard open in $\mathcal{U}$.

Claim 3:

Let X be any integral scheme, $\mathcal{U}, V \subseteq X$ open affine and $f_{\mathcal{U}}: \tilde{\mathcal{U}} \to \mathcal{U}$, $f_V: \tilde{V} \to V$.
Then there is an isomorphism

$\alpha_{\mathcal{U}V}: f_{\mathcal{U}}^{-1}(\mathcal{U} \cap V) \cong f_V^{-1}(\mathcal{U} \cap V)$

Proof:

Cover $\mathcal{U} \cap V$ as in claim 2 with W_i. Then by claim 1 we have isos $f_{\mathcal{U}}^{-1}(W_i) \cong \tilde{W}_i \cong f_V^{-1}(W_i)$.
Now check that these isos coincide over the intersection of two of the W_i: omitted!

Claim 4:

We may glue all $f_{\mathcal{U}}: \tilde{\mathcal{U}} \to \mathcal{U}$ for $\mathcal{U} \subseteq X$ open affine, to a morphism $f_X: \tilde{X} \to X$

Proof:

It remains to check that $\forall\, \mathcal{U}, V, W \subseteq X$ open affine:

$\alpha_{\mathcal{U}W}|_{f_{\mathcal{U}}^{-1}(\mathcal{U} \cap V \cap W)} = \alpha_{VW}|_{\dots} \circ \alpha_{\mathcal{U}V}|_{\dots}$ omitted!

"The Amateur's Proof"

A scheme X is called normal if it is integral and $\forall U \subseteq X$, $\mathcal{O}_X(U)$ is its integral closure in its fraction field (or equivalently if this holds over some open affine cover).

Define the following universal property (UP) for an integral scheme X:

$\tilde{X} \xrightarrow{f_X} X$ is called the normalization if $\forall Y$ normal and $\forall$ ~~f~~ $g: Y \to X$ dominant there is a unique factorization:

$$\begin{tikzcd} Y \arrow[r, dotted, "\exists!"] \arrow[dr, "g"'] & \tilde{X} \arrow[d, "f_X"] \\ & X \end{tikzcd}$$

Claim 1:

If $X = \operatorname{Spec} R$, then $\tilde{X} = \operatorname{Spec} \tilde{R} \to X = \operatorname{Spec} R$ has (UP).

Proof:

Have to check:

$$\begin{tikzcd} R \arrow[r] \arrow[d] & \mathcal{O}_Y(Y) & \text{normal} \arrow[l] \\ \tilde{R} \arrow[ur, dotted, "\exists!"'] \end{tikzcd}$$

$\rightsquigarrow$ of commutative algebra.

Claim 2:

If $U \subseteq X = \operatorname{Spec} R$ is any open subscheme, $\tilde{X} \to X$ the normalization, then $\tilde{U} := U \times_X \tilde{X} \longrightarrow U$ has (UP) for U.

Proof:

let Y normal:

$$\begin{tikzcd} Y \arrow[drr, dashed, bend left, "1"] \arrow[dr, dotted, "2"] \arrow[ddr, bend right, "g"'] & & \\ & U \times_X \tilde{X} \arrow[r] \arrow[d] & \tilde{X} \arrow[d, "f_X"] \\ & U \arrow[r, "j"] & X \end{tikzcd}$$

1 (UP) of X applied to $j \circ g : Y \to X$

2 universal property of fiber product.

Claim 3:

If X is an integral scheme, $\{U_i\}_i$ any open cover of affines. Then the $f_{U_i}: \tilde{U}_i \to U_i$ glue to $f_X: \tilde{X} \to X$

(although we will not prove that $\tilde{X} \to X$ has (UP) for X).

Proof:

$\forall i,j$: $f_{U_i}^{-1}(U_i \cap U_j)$ and $f_{U_j}^{-1}(U_i \cap U_j)$ have by claim 2 both the (UP), hence there is a uniquely determined iso between them; call it α_{ij}.

By uniqueness of the α_{ij}'s we have automatically $\forall i,j,k$:

$$\alpha_{ik}|_{f_{U_i}^{-1}(U_i \cap U_j \cap U_k)} = \alpha_{jk}|_{\dots} \circ \alpha_{ij}|_{\dots}$$

$\Rightarrow$ We may glue the ~~the~~ $f_{U_i} : \tilde{U}_i \to U_i$ along the α_{ij} to $\tilde{X} \to X$.

"The Expert's Proof"

Let $\mathcal{B} = \{U \subseteq X \text{ open affine}\}$. Then we have a $\mathcal{B}$-sheaf of $\mathcal{O}_X$-algebras $\tilde{\mathcal{O}}_X$ defined by $U \mapsto \widetilde{\mathcal{O}_X(U)}$

(use here $(\tilde{R})_f \cong \widetilde{R_f}$ $\forall$ rings R and $f \in R$).

$\rightsquigarrow$ This extends to a sheaf of $\mathcal{O}_X$-algebras $\tilde{\mathcal{O}}_X$ over X.

$\rightsquigarrow$ Define $\tilde{X} = \mathrm{Spec}_{\mathcal{O}_X} \tilde{\mathcal{O}}_X$ relative affine spectrum.

Then $\tilde{X}$ has by ~~the~~ def. all desired properties.