1) Proof 1: (finitetype: obvious; separable: follows from affine)

Claim 1: finite is stable under base change

Proof: Copy the proof for "finite type" (sheet 6, ex. 2+3. (3)).

Claim 2: finite morphisms are closed

Proof:

The property closed is local on the target scheme. Thus wlog

$f: \operatorname{Spec} S \longrightarrow \operatorname{Spec} R$ with $f^\#: R \to S$ finite

Now going up implies $f(V(I)) = V(f^{\#-1}(I))$ $\forall I \subseteq S$ ideal.

Proof 2: Valuation criterion.

Consider:

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r, "\beta"] \arrow[d] & X \arrow[d, "f"] \\ \operatorname{Spec} R \arrow[r, "\alpha"'] \arrow[ur, dashed, "\gamma?"] & Y \end{tikzcd}$$

Let $U = \operatorname{Spec} A \subseteq Y$ any open affine containing $\alpha(\operatorname{Spec} R)$.

Then any $g: \operatorname{Spec} R \to X$ factors as

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & f^{-1}(U) = \operatorname{Spec} B \arrow[d, "f|"] \\ \operatorname{Spec} R \arrow[r, "\alpha"'] \arrow[ur, dashed, "g?"] & U = \operatorname{Spec} A \end{tikzcd}$$

which corresponds to:

$$\begin{tikzcd} K & B \arrow[l, "\beta^\#"'] \\ R \arrow[u, hook] & A \arrow[l, "\alpha^\#"] \arrow[u, "f^\#"'] \end{tikzcd}$$

Thus we have to see that $\forall b \in B$ the element $\beta^\#(b)$ actually lies in $R \subseteq K$. By ~~by~~ finiteness of f $\exists$ integral equation $b^n + \Sigma a_i b^i = 0$ $(a_i \in A)$

$\Rightarrow \beta^\#(b)$ satisfies $x^n + \Sigma \alpha^\#(a_i) x^i = 0$

$\Rightarrow \beta^\#(b)$ is integral over R

But any DVR is normal $\Rightarrow \beta^\#(b) \in R$.

2) Proof 1:

Let $a \in R$ s.th. $\exists\, \mathfrak{p} \in \operatorname{Spec} R$ with $a \in R_{\mathfrak{p}}$ is neither nilpotent nor invertible. Such an element exists because $\dim R \geq 1$.

Now consider

$$\operatorname{Spec} R \times_{\operatorname{Spec} k} \mathbb{A}^1_k \cong \operatorname{Spec} R[t] \longrightarrow \mathbb{A}^1_k = \operatorname{Spec} k[t]$$

Then one can show:

$$V(at-1) \subseteq \operatorname{Spec} R[t] \text{ has image } \mathbb{A}^1_k \setminus \{0\}, \text{ which is not open.}$$

Proof 2:

Claim 1: $\forall n \geq 1$: $\mathbb{A}^n_k \longrightarrow \operatorname{Spec} k$ is not proper ~~if k alg. closed.~~

Proof:

Base-changed to $\mathbb{A}^1_{\bar k} = \operatorname{Spec} \bar{k}[t]$ yields (where $\bar k$ = alg. closure of k)

$\mathbb{A}^{n+1}_{\bar k} \longrightarrow \mathbb{A}^1_{\bar k}$ given by $\bar k[t] \hookrightarrow \bar k[x_1, \dots, x_n, t]$.

Consider the image of $V(x_1 t - 1)$:

$\eta = (0) \subseteq \mathbb{A}^1_{\bar k}$ has preimage $(x_1 t - 1) \subseteq \bar k[x_1, \dots, x_n, t]$

$(t - \lambda) \in \mathbb{A}^1_{\bar k}$ $(\lambda \in \bar k \setminus 0)$ has preimage $(t - \lambda, x_1 - \lambda^{-1}) \subseteq \bar k[x_1, \dots, x_n, t]$.

$(t) \in \mathbb{A}^1_{\bar k}$ has <u>no</u> preimage as any such point $\mathfrak{p} \subseteq \bar k[x_1, \dots, x_n, t]$ has to satisfy $\mathfrak{p} \supseteq (t, x_1 t - 1) = (1)$. $\lightning$

$\Rightarrow \mathbb{A}^{n+1}_{\bar k} \longrightarrow \mathbb{A}^1_{\bar k}$ not closed.

Claim 2: If R is integral $\overset{+\dim R \geq 1}{\vee}$, $\operatorname{Spec} R \longrightarrow \operatorname{Spec} k$ is not proper.

Proof:

By Noether-normalization there is a morphism

$k[x_1, \dots, x_n] \hookrightarrow R$ s.th. R finite $k[x_1, \dots, x_n]$-module.

$$\Rightarrow \exists \quad \begin{tikzcd} \operatorname{Spec} R \arrow[r, two heads, "\text{finite}", "\text{surj.}"'] \arrow[d] & \mathbb{A}^n_k \arrow[dl] \\ \operatorname{Spec} k & \end{tikzcd} \quad \overset{\text{base-}}{\underset{\text{change}}{\rightsquigarrow}} \quad \begin{tikzcd} \operatorname{Spec} R \times \mathbb{A}^1_{\bar k} \arrow[r, two heads, "\text{finite}", "\text{surj.}"'] \arrow[d] & \mathbb{A}^n \times \mathbb{A}^1_{\bar k} \arrow[dl] \\ \mathbb{A}^1_{\bar k} & \end{tikzcd}$$

By the previous claim $\exists\, Z \subseteq \mathbb{A}^n \times \mathbb{A}^1_k$ with non-closed image.

$\Rightarrow$ Its preimage in $\operatorname{Spec} R \times \mathbb{A}^1_{\bar k}$ ~~the~~ is closed and has the same image in $\mathbb{A}^1_{\bar k}$.

$\Longrightarrow \operatorname{Spec} R \longrightarrow \operatorname{Spec} k$ not proper.

Claim 3: ~~Any~~ Spec $R \to$ Spec k not proper $\forall R$ with $\dim R \geq 1$.

Proof:

reduced.

$\exists$ irreducible component $Z = \operatorname{Spec} R' \subseteq \operatorname{Spec} R$ which has $\dim \geq 1$.

$\Rightarrow$ $Z \longrightarrow \operatorname{Spec} k$ not proper by Claim 2.

But $Z \hookrightarrow \operatorname{Spec} R$ is proper (it is a closed immersion!). Thus $\operatorname{Spec} R \to \operatorname{Spec} k$ cannot be proper by ex. 3+4. (2).

3+4.)

(1)
$$\begin{tikzcd} \operatorname{Spec} K \arrow[r, "\beta"] \arrow[d, "j"'] & X \times_Y Y' \arrow[r, "pr_1"] \arrow[d, "f'"'] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\ \operatorname{Spec} R \arrow[r, "\alpha"] & Y' \arrow[r] & Y \end{tikzcd}$$

<u>separated</u>:

Let $h_1, h_2 : \operatorname{Spec} R \to X \times_Y Y'$. Then by val. criteria for f we have $pr_1 \circ h_1 = pr_2 \circ h_2$. Furthermore by def. of h_i we have $f' \circ h_1 = \alpha = f' \circ h_2$

$\Rightarrow$ Univ. prop. of $X \times_Y Y'$ implies: $h_1 = h_2$.

<u>proper</u>:

As f proper: $\exists \tilde{h} : \operatorname{Spec} R \to X$. Then define

$h = (\tilde{h}, \alpha) : \operatorname{Spec} R \to X \times_Y Y'$.

- $\alpha = f' \circ h$: by construction
- $\beta = h \circ j$: This follows from the univ. prop. of $X \times_Y Y'$ by

$$pr_1 \circ \beta \overset{\text{def. of } \tilde{h}}{=} \tilde{h} \circ j = pr_1 \circ h \circ j$$
$$f' \circ \beta = \alpha \circ j = f' \circ h \circ j.$$

This method works for (2) and (3) as well. Nevertheless we will prove them by checking the definitions directly (or at least give sketches)

(2) Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ with f, g both separated. Then:

$$\begin{tikzcd} X \arrow[d, "\Delta_{X/Z}"'] & \subseteq & X \times_Y Y \arrow[d, "\Delta_{X \times_Y Y/Z}"'] & \cong & X \arrow[d, "\Delta_{X/Y}"'] & \times_Y & Y \arrow[d, "\Delta_{Y/Z}"] \\ X \times_Z X & \cong & (X \times_Y Y) \times_Z (Y \times_Y X) & \cong & X \times_Y (Y \times_Z Y) \times_Y X & \subseteq & (X \times_Y X) \times_Y (Y \times_Z Y) \end{tikzcd}$$

is commutative.

Then:

$$\begin{tikzcd}
X\times_Y Y \arrow[r,"pr_1"] \arrow[d,"{(\Delta_{X/Y},id_Y)}"'] \arrow[dr, phantom, "\square"] \arrow[ddd, bend right=60, "{(\Delta_{X/Y},\Delta_{Y/Z})}"'] & X \arrow[d,"\Delta_{X/Y}"] \\
(X\times_Y X)\times_Y Y \arrow[r,"pr_1"] \arrow[dr,"pr_2"] \arrow[dd,"{(id,\Delta_{Y/Z})}"'] & X\times_Y X \\
{} \arrow[r, phantom, "\square"] & Y \arrow[d,"\Delta_{Y/Z}"] \\
(X\times_Y X)\times_Y(Y\times_Z Y) \arrow[r,"pr_2"] & Y\times_Z Y
\end{tikzcd}$$

$\Rightarrow (\Delta_{X/Y}, id_Y)$ and $(id, \Delta_{Y/Z})$ are closed immersions

$\Rightarrow (\Delta_{X/Y}, \Delta_{Y/Z})$ closed immersion

$\Rightarrow \Delta_{X/Z}$ closed immersion.

Let now $X \xrightarrow{f} Y \xrightarrow{g} Z$ with f, g proper. Let $S \to Z$ be any morph.

$$\begin{tikzcd}
X\times_Z S \arrow[ddrr, "pr_2"'] & \cong (X\times_Y Y)\times_Z S \cong & X\times_Y(Y\times_Z S) \arrow[r] \arrow[d,"pr_2"'] \arrow[dr, phantom, "\square"] & X \arrow[d,"f"] \\
& \circlearrowleft & Y\times_Z S \arrow[r] \arrow[d,"pr_2"'] \arrow[dr, phantom, "\square"] & Y \arrow[d,"g"] \\
& & S \arrow[r] & Z
\end{tikzcd}$$

$\Rightarrow pr_2 : X\times_Y(Y\times_Z S) \to Y\times_Z S$ closed & $pr_2 : Y\times_Z S \to S$ closed

$\Rightarrow pr_2 : X\times_Z S \to S$ closed

$\Rightarrow$ ~~$f\circ g$~~ $g\circ f$ univ. closed.

(3) Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ with $g\circ f$ separated.

First note that $pr_1 : X\times_Y X \to X$ and $pr_2 : X\times_Y X \to X$ (the two projections) define a morphism $X\times_Y X \xrightarrow{i} X\times_Z X$. Then we have a comm. diagram:

$$\begin{tikzcd}
X \arrow[r,"\Delta_{X/Y}"] \arrow[dr,"\Delta_{X/Z}"'] & X\times_Y X \arrow[d,"i"] \arrow[dl, phantom, "\circlearrowleft"] \\
& X\times_Z X
\end{tikzcd}$$

because $i\circ\Delta_{X/Y}$ and $\Delta_{X/Z}$ agree after projection to each X-factor.

Write i as:

$$\begin{tikzcd}
X\times_Y X \arrow[rr, equal] \arrow[d,"i"'] & \circlearrowleft & X\times_Y Y\times_Y X \arrow[d,"{(id_X,\Delta_{Y/Z},id_X)}"] \\
X\times_Z X \cong (X\times_Y Y)\times_Z(Y\times_Y X) & \cong & X\times_Y(Y\times_Z Y)\times_Y X
\end{tikzcd}$$

$\Rightarrow i$ is injective $\Rightarrow \mathrm{im}(\Delta_{X/Y}) = i^{-1}(\underbrace{\mathrm{im}(\Delta_{X/Z})}_{\text{closed}})$ is closed.

Let now $X \xrightarrow{f} Y \xrightarrow{g} Z$ with $g \circ f$ proper <u>and g separated</u> !!
Let $S \to Y$ be any morphism and define $i : X \times_Y S \to X \times_Z S$ as before. Then we have a comm. diagr.

$$\begin{tikzcd}
 & X \times_Y S \arrow[r] \arrow[dl, "i"'] \arrow[d, "f'"] & X \arrow[d, "f"] \\
X \times_Z S \arrow[r, "pr_2"] & S \arrow[r] & Y
\end{tikzcd}$$

By properness of $g \circ f : X \to Z$ the morph. $pr_2 : X \times_Z S \to S$ is closed.
Hence we are reduced to prove the following

Claim:

$X \times_Y S \xrightarrow{i} X \times_Z S$ is a closed immersion if $Y \to S$ is separated.

Proof:

Write as above:

$$\begin{tikzcd}
X \times_Y S \arrow[d, "i"'] \arrow[rr, phantom, "\cong"] & & X \times_Y Y \times_Y S \arrow[d, "(id_X \times \Delta_{Y/Z},\, id_S)"] \\
X \times_Z S \cong (X \times_Y Y) \times_Z (Y \times_Y S) \arrow[rr, phantom, "\cong"] & & X \times_Y (Y \times_Z Y) \times_Y S
\end{tikzcd}$$

But $\Delta_{Y/Z}$ is a closed immersion and closed immersions are stable under base change

$\Rightarrow i : X \times_Y S \to X \times_Z S$ is a closed immersion.

WARNING:

The statement (3) for proper is wrong if g is not separated:
Consider $\mathbb{A}^1 \xrightarrow{f} \mathbb{A}^1 \amalg_{\mathbb{A}^1 \setminus 0} \mathbb{A}^1 \xrightarrow{pr} \mathbb{A}^1$ where f is the inclusion into the first $\mathbb{A}^1$ and pr denotes the identification of the double point.
Then $pr \circ f = id_{\mathbb{A}^1}$ is proper, but f is not:

$$\begin{tikzcd}
\operatorname{Spec} k(x) \arrow[rr, hook] \arrow[d] & & \mathbb{A}^1 \arrow[d, "f"] \\
\operatorname{Spec} k[x]_{(x)} \arrow[r, hook] \arrow[urr, dashed, "h?"] & \operatorname{Spec} k[x] \arrow[r, hook] & \mathbb{A}^1 \amalg_{\mathbb{A}^1 \setminus 0} \mathbb{A}^1
\end{tikzcd}$$

inclusion into second $\mathbb{A}^1$

But there cannot be some h, because the image of $\operatorname{Spec} k[x]_{(x)}$ is not contained in the image of f!