

1) Proof 1: (finite type: obvious; separable: follows from affine)

Claim 1: finite is stable under base change

Proof: Copy the proof for "finite type" (sheet 6, ex. 2.13. (3)).

Claim 2: finite morphisms are closed

Proof:

The property closed is local on the target scheme. Thus wlog

$f: \text{Spec } S \rightarrow \text{Spec } R$ with $f^\#: R \rightarrow S$ finite

Now going up implies $f(V(I)) = V(f^{\#-1}(I)) \forall I \subseteq S$ ideal.

Proof 2: Valuation criterion

Consider:

$$\begin{array}{ccc} \text{Spec } K & \xrightarrow{\beta} & X \\ \downarrow & \nearrow \beta^\# & \downarrow f \\ \text{Spec } R & \xrightarrow{\alpha} & Y \end{array}$$

Let $U = \text{Spec } A \subseteq Y$ any open affine containing $\alpha(\text{Spec } R)$.

Then any $g: \text{Spec } R \rightarrow X$ factors as

$$\begin{array}{ccc} \text{Spec } K & \longrightarrow & f^{-1}(U) = \text{Spec } B \\ \downarrow & \nearrow g^\# & \downarrow f|_{f^{-1}(U)} \\ \text{Spec } R & \xrightarrow{\alpha} & U = \text{Spec } A \end{array}$$

which corresponds to:

$$\begin{array}{ccc} K & \xleftarrow{\beta^\#} & B \\ \uparrow & & \uparrow f^\# \\ R & \xleftarrow{\alpha^\#} & A \end{array}$$

Thus we have to see that $\forall b \in B$ the element $\beta^\#(b)$ actually lies in $R \subseteq K$. By the finiteness of f , \exists integral equation $b^n + \sum a_i b^i = 0$ ($a_i \in A$)

$\Rightarrow \beta^\#(b)$ satisfies $x^n + \sum \alpha^\#(a_i) x^i = 0$

$\Rightarrow \beta^\#(b)$ is integral over R

But any DVR is normal $\Rightarrow \beta^\#(b) \in R$.

2) Proof 1:

Let $\alpha \in R$ s.t. $\exists p \in \text{Spec } R$ with $\alpha \in R_p$ is neither nilpotent nor invertible. Such an element exists because $\dim R \geq 1$.

Now consider

$$\text{Spec } R \times_{\text{Spec } k} A_k^1 \cong \text{Spec } R[t] \longrightarrow A_k^1 = \text{Spec } k[t]$$

Then one can show:

$V(\alpha t - 1) \in \text{Spec } R[t]$ has image $A_k^1 \setminus \{0\}$, which is not open.

Proof 2:

Claim 1: $\forall n \geq 1: A_k^n \rightarrow \text{Spec } k$ is not proper ~~if k is alg. closed.~~

Proof:

base-changed to $A_{\bar{k}}^1 = \text{Spec } \bar{k}[t]$ yields (where \bar{k} = alg. closure of k)

$$A_{\bar{k}}^{n+1} \rightarrow A_{\bar{k}}^1 \text{ given by } \bar{k}[t] \hookrightarrow \bar{k}[x_1, \dots, x_n, t].$$

Consider the image of $V(x_1 t - 1)$:

$$\eta = (0) \in A_{\bar{k}}^1 \text{ has preimage } (x_1 t - 1) \in \bar{k}[x_1, \dots, x_n, t]$$

$$(t - \lambda) \in A_{\bar{k}}^1 (\lambda \in \bar{k} \setminus \{0\}) \text{ has preimage } (t - \lambda, x_1 - \lambda^{-1}) \in \bar{k}[x_1, \dots, x_n, t].$$

$$(t) \in A_{\bar{k}}^1 \text{ has no preimage as any such point } p \in \bar{k}[x_1, \dots, x_n, t]$$

$$\text{has to satisfy } p \supseteq (t, x_1 t - 1) = (1). \quad \Downarrow$$

$$\Rightarrow A_{\bar{k}}^{n+1} \rightarrow A_{\bar{k}}^1 \text{ not closed.}$$

Claim 2: If R is integral ^{$\dim R \geq 1$} , $\text{Spec } R \rightarrow \text{Spec } k$ is not proper.

Proof:

By Noether-normalization there is a morphism

$$k[x_1, \dots, x_n] \hookrightarrow R \text{ s.t. } R \text{ finite } k[x_1, \dots, x_n]\text{-module.}$$

$$\Rightarrow \begin{array}{ccc} \text{Spec } R & \xrightarrow{\text{finite surj.}} & A_k^n \\ \downarrow & \swarrow & \downarrow \\ \text{Spec } k & & A_k^1 \end{array} \quad \begin{array}{ccc} \text{Spec } R \times A_k^1 & \xrightarrow{\text{finite surj.}} & A_k^n \times A_k^1 \\ \downarrow & \swarrow & \downarrow \\ A_k^1 & & A_k^1 \end{array}$$

base-change

By the previous claim $\exists Z \subseteq A_k^n \times A_k^1$ with non-closed image.

\Rightarrow Its preimage in $\text{Spec } R \times A_k^1$ is closed and has the same image in A_k^1 .

$\Rightarrow \text{Spec } R \rightarrow \text{Spec } k$ not proper.

Claim 3: ~~Any~~ $\text{Spec } R \rightarrow \text{Spec } k$ not proper $\forall R$ with $\dim R \geq 1$.

Proof:

\exists irreducible component $Z = \text{Spec } R' \subseteq \text{Spec } R$ which has $\dim \geq 1$.
 $\Rightarrow Z \rightarrow \text{Spec } k$ not proper by claim 2.

But $Z \hookrightarrow \text{Spec } R$ is proper (it is a closed immersion!). Thus $\text{Spec } R \rightarrow \text{Spec } k$ cannot be proper by ex. 3+4. (2).

3+4.)

$$(1) \begin{array}{ccccc} \text{Spec } K & \xrightarrow{\beta} & X \times_Y Y' & \xrightarrow{pr_1} & X \\ j \downarrow & & \downarrow f' & \square & \downarrow f \\ \text{Spec } R & \xrightarrow{\alpha} & Y' & \longrightarrow & Y \end{array}$$

separated:

Let $h_1, h_2: \text{Spec } R \rightarrow X \times_Y Y'$. Then by val. criteria for f we have $pr_1 \circ h_1 = pr_1 \circ h_2$. Furthermore by def. of h_i we have $f' \circ h_1 = \alpha = f' \circ h_2$
 \Rightarrow Univ. prop. of $X \times_Y Y'$ implies: $h_1 = h_2$.

proper:

As f proper: $\exists \tilde{h}: \text{Spec } R \rightarrow X$. Then define

$$h = (\tilde{h}, \alpha): \text{Spec } R \rightarrow X \times_Y Y'$$

• $\alpha = f' \circ h$: by construction

• $\beta = h \circ j$: This follows from the univ. prop. of $X \times_Y Y'$ by

$$pr_1 \circ \beta = \tilde{h} \circ j = pr_1 \circ h \circ j$$

$$f' \circ \beta = \alpha \circ j = f' \circ h \circ j.$$

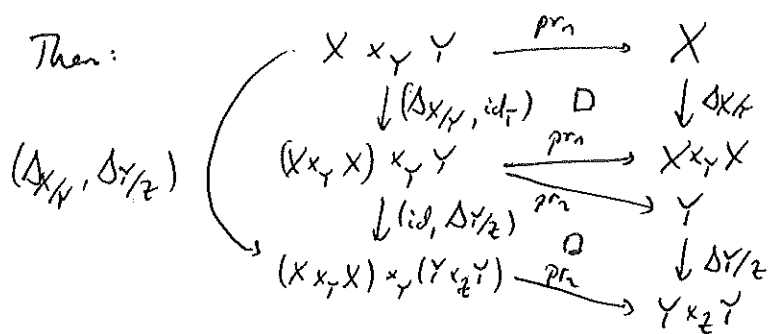
This method works for (2) and (3) as well. Nevertheless we will prove them by checking the definitions directly (or at least give sketches)

(2) Let $X \xrightarrow{f} Z, Y \xrightarrow{g} Z$ with f, g both separated. Then:

$$\begin{array}{ccc} X & \subseteq & X \times_Y Y \\ \Delta_{X/Z} \downarrow & & \Delta_{X \times_Y Y / Z} \downarrow \\ X \times_Z X & \cong & (X \times_Y Y) \times_Z (Y \times_Y X) \cong X \times_Y (Y \times_Z Y) \times_Y X \cong (X \times_Y X) \times_Y (Y \times_Z Y) \end{array}$$

is commutative.

Then:

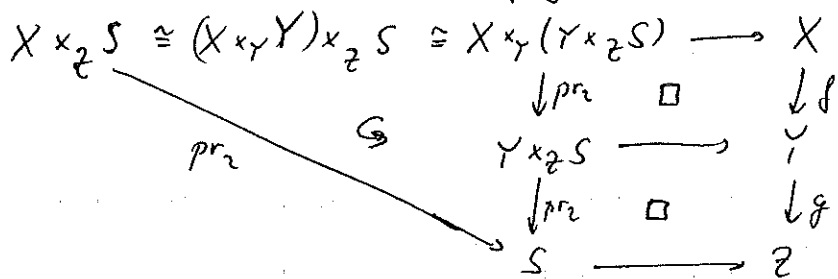


$\Rightarrow (\Delta_{X/Y}, \text{id}_Y)$ and $(\text{id}, \Delta_{Y/Z})$ are closed immersions

$\Rightarrow (\Delta_{X/Y}, \Delta_{Y/Z})$ closed immersion

$\Rightarrow \Delta_{X/Z}$ closed immersion.

Let now $X \xrightarrow{f} Y \xrightarrow{g} Z$ with f, g proper. Let $S \rightarrow Z$ be any morph.



$\Rightarrow \text{pr}_2: X \times_Y (Y \times_Z S) \rightarrow Y \times_Z S$ closed & $\text{pr}_2: Y \times_Z S \rightarrow S$ closed

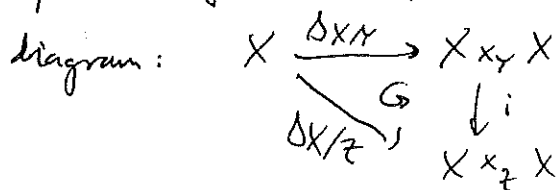
$\Rightarrow \text{pr}_2: X \times_Z S \rightarrow S$ closed

$\Rightarrow f \circ g$ of univ. closed.

(3) Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ with g of separated.

First note that $\text{pr}_1: X \times_Y X \rightarrow X$ and $\text{pr}_2: X \times_Y X \rightarrow X$ (the two projections)

define a morphism $X \times_Y X \xrightarrow{i} X \times_Z X$. Then we have a comm.



because $i \circ \Delta_{X/Y}$ and $\Delta_{X/Z}$ agree after projection to each X -factor.

Write i as:

$$\begin{array}{ccc}
 X \times_Y X & \xrightarrow{\cong} & X \times_Y Y \times_Y X \\
 \downarrow i & \searrow \cong & \downarrow (\text{id}_X, \Delta_{Y/Z}, \text{id}_X) \\
 X \times_Z X & \cong & (X \times_Y Y) \times_Z (Y \times_Y X) \cong X \times_Y (Y \times_Z Y) \times_Y X
 \end{array}$$

$\Rightarrow i$ is injective $\Rightarrow \text{im}(\Delta_{X/Z}) = i^{-1}(\underbrace{\text{im}(\Delta_{X/Z})}_{\text{closed}})$ is closed.

Let now $X \xrightarrow{f} Y \xrightarrow{g} Z$ with $g \circ f$ proper and g separated !!

Let $S \rightarrow Y$ be any morphism and define $i: X \times_Y S \rightarrow X \times_Z S$ as before. Then we have a comm. diagr.

$$\begin{array}{ccccc} & & X \times_Y S & \longrightarrow & X \\ & i \swarrow & & \downarrow f' & \downarrow f \\ X \times_Z S & \xrightarrow{pr_2} & S & \longrightarrow & Y \end{array}$$

By properness of $g \circ f: X \rightarrow Z$ the morph. $pr_2: X \times_Z S \rightarrow S$ is closed. Hence we are reduced to prove the following

Claim:

$X \times_Y S \xrightarrow{i} X \times_Z S$ is a closed immersion if $Y \rightarrow S$ is separated.

Proof:

Write as above:

$$\begin{array}{ccc} X \times_Y S & \xrightarrow{\cong} & X \times_Y Y \times_Y S \\ \downarrow i & & \downarrow (id_X \times \Delta_{Y/Z}, id_S) \\ X \times_Z S & \cong (X \times_Y Y) \times_Z (Y \times_Y S) & \cong X \times_Y (Y \times_Z Y) \times_Y S \end{array}$$

But $\Delta_{Y/Z}$ is a closed immersion and closed immersions are stable under base change

$\Rightarrow i: X \times_Y S \rightarrow X \times_Z S$ is a closed immersion.

WARNING:

The statement (3) for proper is wrong if g is not separated:

Consider $A^1 \xrightarrow{f} A^1 \sqcup_{A^1 \setminus \{0\}} A^1 \xrightarrow{pr} A^1$ where f is the inclusion into the first A^1 and pr denotes the identification of the double point.

Then $pr \circ f = id_{A^1}$ is proper, but f is not:

$$\begin{array}{ccc} \text{Spec } k[x] & \xrightarrow{\quad} & A^1 \\ \downarrow & \nearrow h' & \downarrow f \\ \text{Spec } k[x]_{(x)} & \xrightarrow{\quad} & A^1 \sqcup_{A^1 \setminus \{0\}} A^1 \\ & \uparrow & \text{inclusion into second } A^1 \end{array}$$

But there cannot be some h , because the image of $\text{Spec } k[x]_{(x)}$ is not contained in the image of f !