Ex 1

(1) R comes with

- comultiplication $\Delta : R \longrightarrow R \otimes_k R$ ($\hat{=}$ group multiplication)
- antipode : $\iota : R \longrightarrow R$ ($\hat{=}$ inverse elements)
- counit : $e : R \longrightarrow k$ ($\hat{=}$ unit element)

such that:

$$\begin{tikzcd}
R \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & R \otimes_k R \arrow[d, "id \otimes \Delta"] \\
R \otimes_k R \arrow[r, "\Delta \otimes id"] & R \otimes_k R \otimes_k R
\end{tikzcd}
\qquad
\begin{tikzcd}
R \arrow[r, "\Delta"] \arrow[d, "\Delta"'] \arrow[dr, "id"] & R \otimes_k R \arrow[d, "e \otimes id"] \\
R \otimes_k R \arrow[r, "id \otimes e"'] & R
\end{tikzcd}$$

$$\begin{tikzcd}
R \otimes_k R \arrow[d, "id \otimes \iota"'] & R \arrow[l, "\Delta"'] \arrow[r, "\Delta"] \arrow[d, "e"] & R \otimes_k R \arrow[d, "\iota \otimes id"] \\
R \otimes_k R \arrow[r, "mult."] & R & R \otimes_k R \arrow[l, "mult."'] \\
& k \arrow[u, phantom, ""]
\end{tikzcd}$$

(2) Recall: $\mu_{n,k} = \operatorname{Spec} R$ with $R = k[x]/(x^n - 1)$

$\cancel{\Delta_\mu}\, m_\mu : R \otimes R \longrightarrow R \quad , \quad x^i \otimes x^j \longmapsto x^{i+j}$

$\eta_\mu : k \hookrightarrow R \quad , \quad 1 \longmapsto 1$

$\Delta_\mu : R \longrightarrow R \otimes R \quad , \quad x^i \longmapsto x^i \otimes x^i$

$\iota_\mu : R \longrightarrow R \quad , \quad x^i \longmapsto x^{-i}$

$e_\mu : R \longrightarrow k \quad , \quad x^i \longmapsto 1$

$$\left[\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{else} \end{cases} \right]$$

and: $\mathbb{Z}/n\mathbb{Z}_k = \operatorname{Spec} \bigoplus_{i \in \mathbb{Z}/n\mathbb{Z}} k e_i =: \operatorname{Spec} S$

$m_{\mathbb{Z}} : (\bigoplus_i k e_i) \otimes (\bigoplus_j k e_j) \longrightarrow \bigoplus_i k e_i \; ; \; e_i \otimes e_j \longmapsto \delta_{ij} e_i$

$\eta_{\mathbb{Z}} : k \hookrightarrow \bigoplus_i k e_i \; ; \; 1 \longmapsto e_0$

$\Delta_{\mathbb{Z}} : \bigoplus_i k e_i \longrightarrow (\bigoplus_i k e_i) \otimes (\bigoplus_j k e_j) \; ; \; e_i \longmapsto \sum_{j \in \mathbb{Z}/n\mathbb{Z}} e_j \otimes e_{i-j}$

$\iota_{\mathbb{Z}} : \bigoplus_i k e_i \longrightarrow \bigoplus_i k e_i \; ; \; e_i \longmapsto e_{-i}$

$e_{\mathbb{Z}} : \bigoplus_i k e_i \longrightarrow k \; , \; e_i \longmapsto \delta_{i0}$.

Then define the isomorphism of vector spaces:

$$f : R \longrightarrow S^* = \operatorname{Hom}(S, k)$$
$$x^i \longmapsto (e_a \mapsto \delta_{ia})$$

Then we check:

$$\begin{tikzcd} R \arrow[r, "\Delta_\mu"] \arrow[d, "f"'] & R\otimes R \arrow[d, "f\otimes f"] \\ S^* \arrow[r, "m_{\mathbb{Z}}^*"] & S^*\otimes S^* \end{tikzcd} \qquad \begin{tikzcd} x^i \arrow[rr, mapsto] \arrow[d, mapsto] & & x^i\otimes x^i \arrow[d, mapsto] \\ (e_a\mapsto \delta_{ia}) \arrow[r, mapsto] & (e_a\otimes e_b\mapsto \delta_{ab}\cdot\delta_{ia}) & = (e_a\otimes e_b \mapsto \delta_{ia}\cdot\delta_{ib}). \end{tikzcd}$$

$$\begin{tikzcd} R\otimes R \arrow[r, "m_\mu"] \arrow[d, "f\otimes f"'] & R \arrow[d, "f"] \\ S^*\otimes S^* \arrow[r, "\Delta_{\mathbb{Z}}^*"] & S^* \end{tikzcd} \qquad \begin{tikzcd} x^i\otimes x^j \arrow[rr, mapsto] \arrow[d, mapsto] & & x^{i+j} \arrow[d, mapsto] \\ (e_a\otimes e_b\mapsto \delta_{ia}\delta_{jb}) \arrow[r, mapsto] & (e_a\mapsto \sum_{b\in\mathbb{Z}/n\mathbb{Z}}\delta_{ib}\cdot\delta_{j(a-b)}) \arrow[d, equal] & (e_a\mapsto\delta_{(i+j)a}) \arrow[dl, equal] \\ & (e_a\mapsto \delta_{j(a-i)}) & \end{tikzcd}$$

$$\begin{tikzcd} k \arrow[r, "\eta_\mu"] \arrow[d, "\cong"'] & R \arrow[d, "f"] \\ k^* \arrow[r, "e_{\mathbb{Z}}^*"] & S^* \end{tikzcd} \qquad \begin{tikzcd} 1 \arrow[rr, mapsto] \arrow[d, mapsto] & & 1 = x^0 \arrow[d, mapsto] \\ (1\mapsto 1) \arrow[r, mapsto] & (e_a\mapsto\delta_{a0}) & = (e_a\mapsto\delta_{a0}) \end{tikzcd}$$

$$\begin{tikzcd} R \arrow[r, "e_\mu"] \arrow[d, "f"'] & k \arrow[d, "\cong"] \\ S^* \arrow[r, "\eta_{\mathbb{Z}}^*"] & k^* \end{tikzcd} \qquad \begin{tikzcd} x^i \arrow[rr, mapsto] \arrow[d, mapsto] & & 1 \arrow[d, mapsto] \\ (e_a\mapsto\delta_{ia}) \arrow[r, mapsto] & (1\mapsto \sum_a \delta_{ia}) & = (1\mapsto 1) \end{tikzcd}$$

$$\begin{tikzcd} R \arrow[r, "i_\mu"] \arrow[d, "f"'] & R \arrow[d, "f"] \\ S^* \arrow[r, "i_{\mathbb{Z}}^*"] & S^* \end{tikzcd} \qquad \begin{tikzcd} x^i \arrow[rr, mapsto] \arrow[d, mapsto] & & x^{-i} \arrow[d, mapsto] \\ (e_a\mapsto\delta_{ia}) \arrow[r, mapsto] & (e_a\mapsto\delta_{i(-a)}) & = (e_a\mapsto\delta_{(-i)a}) \end{tikzcd}$$

$\Longrightarrow$ f induces an isomorphism of group schemes $\mu_{n,k} \cong (\underline{\mathbb{Z}/n\mathbb{Z}}_k)^D$.

Dualizing all diagrams above (and using the canonical iso $S^{**}\cong S$) gives:

f^* induces an isomorphism $\underline{\mathbb{Z}/n\mathbb{Z}}_k \cong \mu_{n,k}{}^D$.

Ex 2

(1) $\left\{ \begin{array}{c} \operatorname{Spec} k \longrightarrow \mathbb{G}_{a,k} \\ \searrow \quad \swarrow \\ \operatorname{Spec} k \end{array} \right\} \cong \left\{ \begin{array}{c} k\text{-linear maps} \\ k[x] \longrightarrow k \end{array} \right\} \cong \left\{ \text{elements of } k \right\}$

$$k[x] \xrightarrow{f} k[x]/(x-\lambda) \cong k. \quad \longmapsto \quad f(x)$$
$$\longleftarrow\!\!\shortmid \quad \lambda$$

Let now $\lambda, \lambda' \in k$ with corresp. points $p_\lambda, p_{\lambda'} : \operatorname{Spec} k \longrightarrow \mathbb{G}_{a,k}$.
Then $p_\lambda \cdot p_{\lambda'} : \operatorname{Spec} k \longrightarrow \mathbb{G}_{a,k}$ is given by:

$$\operatorname{Spec} k \xrightarrow{(p_\lambda, p_{\lambda'})} \mathbb{G}_a \times_{\operatorname{Spec} k} \mathbb{G}_a \xrightarrow{\text{mult.}} \mathbb{G}_a$$

On algebras:
$$k[x] \xrightarrow{\Delta} k[x] \otimes_k k[x] \xrightarrow{p_\lambda^\# \otimes p_{\lambda'}^\#} k \otimes_k k \cong k$$

$$\cancel{x \longmapsto 1 \otimes x + x \otimes 1 \longmapsto \lambda \otimes \lambda' = \lambda \cdot}$$

$$x \longmapsto 1 \otimes x + x \otimes 1 \longmapsto 1 \otimes \lambda' + \lambda \otimes 1 = \lambda + \lambda'$$

$$\Longrightarrow p_\lambda \cdot p_{\lambda'} = p_{\lambda + \lambda'}.$$

(2) $\left\{ \begin{array}{c} \operatorname{Spec} k \longrightarrow \mathbb{G}_m \\ \searrow \quad \swarrow \\ \operatorname{Spec} k \end{array} \right\} \cong \left\{ \begin{array}{c} k\text{-linear maps} \\ k[x^{\pm 1}] \longrightarrow k \end{array} \right\} \cong \left\{ \text{elements of } k^\times \right\}$

$$k[x^{\pm 1}] \xrightarrow{f} k[x]/(x-\lambda) \cong k. \quad \longmapsto \quad f(x)$$
$$\longleftarrow\!\!\shortmid \quad \lambda$$

Let again $\lambda, \lambda' \in k^\times$ with corresp. points $p_\lambda, p_{\lambda'} : \operatorname{Spec} k \longrightarrow \mathbb{G}_{m,k}$
Then $p_\lambda \cdot p_{\lambda'} : \operatorname{Spec} k \longrightarrow \mathbb{G}_{m,k}$ is given by

$$\operatorname{Spec} k \xrightarrow{(p_\lambda, p_{\lambda'})} \mathbb{G}_m \times_{\operatorname{Spec} k} \mathbb{G}_m \xrightarrow{\text{mult}} \mathbb{G}_m$$

On algebras
$$k[x^{\pm 1}] \xrightarrow{\Delta} k[x^{\pm 1}] \otimes k[x^{\pm 1}] \xrightarrow{p_\lambda^\# \otimes p_{\lambda'}^\#} k \otimes_k k \cong k$$

$$x \longmapsto x \otimes x \longmapsto \lambda \otimes \lambda' = \lambda \cdot \lambda'$$

$$\Longrightarrow p_\lambda \cdot p_{\lambda'} = p_{\lambda \cdot \lambda'}.$$

Ex 3

(1) Assume we have $\mathbb{G}_{m,R} \longrightarrow \mathbb{G}_{a,R}$ given by a morphism between the corresponding Hopf-algebras

$$f : R[x] \longrightarrow R[x^{\pm 1}] \quad \text{with } f(x) = \sum_{i \in \mathbb{Z}} \lambda_i x^i$$

Then we have:

$$\begin{tikzcd}
R[x] \arrow[r, "f"] \arrow[d, "e_{\mathbb{G}_a}"'] \arrow[dr, phantom, "\circlearrowleft"] & R[x^{\pm 1}] \arrow[d, "e_{\mathbb{G}_m}"] & x \arrow[r, mapsto] \arrow[d, mapsto] & \sum \lambda_i x^i \arrow[d, mapsto] \\
R \arrow[r, equal] & R & 0 \arrow[r, equal] & \sum \lambda_i
\end{tikzcd}$$

$$\Longrightarrow \sum_i \lambda_i = 0 \qquad (*)$$

and:

$$\begin{tikzcd}
R[x] \arrow[r, "f"] \arrow[d, "\Delta_{\mathbb{G}_a}"'] & R[x^{\pm 1}] \arrow[d] & x \arrow[r, mapsto] \arrow[d, mapsto] & \sum \lambda_i x^i \arrow[d, mapsto] \\
R[x] \otimes_R R[x] \arrow[r, "f \otimes f"] & R[x^{\pm 1}] \otimes_R R[x^{\pm 1}] & 1 \otimes x + x \otimes 1 \arrow[r, mapsto] & \sum_{i,j} \lambda_i \lambda_j x^i \otimes x^j = \sum_i \lambda_i 1 \otimes x^i + \sum_i \lambda_i x^i \otimes 1
\end{tikzcd}$$

Comparing coeff. in front of $x^i \otimes 1$ yields

$$\begin{cases} \lambda_i \lambda_0 = \lambda_i & \text{if } i \neq 0 \\ \lambda_0^2 = 2\lambda_0 \end{cases} \qquad (**)$$

With $(*)$ we have

$$0 = \sum_i \lambda_i = \sum_{i \neq 0} \lambda_i \lambda_0 + \lambda_0^2 - \lambda_0 = \lambda_0 \Big(\sum_i \lambda_i\Big) - \lambda_0 = \lambda_0$$

$\Longrightarrow$ With $(**)$: $\lambda_i = 0 \quad \forall i \in \mathbb{Z}$

Hence f is given by:

$$R[x] \xrightarrow[e_{\mathbb{G}_a}]{\text{mod } x} R \hookrightarrow R[x^{\pm 1}]$$

or in terms of group schemes:

$$\mathbb{G}_{m,R} \longrightarrow \underline{1}_R \longrightarrow \mathbb{G}_{a,R}$$

where $\underline{1}_R$ denotes the constant group scheme given by the trivial group.

(2) Let R be reduced and $f: \mathbb{G}_{a,R} \to \mathbb{G}_{m,R}$ given by

$$f: R[x^{\pm 1}] \longrightarrow R[x], \quad f(x) = \sum_{i\geq 0} \lambda_i x^i$$

Then:

$$\begin{tikzcd} R[x^{\pm1}] \arrow[r, "f"] \arrow[d, "\Delta_{\mathbb{G}_m}"'] & R[x] \arrow[d, "\Delta_{\mathbb{G}_a}"] \\ R[x^{\pm1}]\otimes R[x^{\pm1}] \arrow[r, "f\otimes f"] & R[x]\otimes R[x] \end{tikzcd} \qquad \begin{tikzcd} x \arrow[r, mapsto] \arrow[d, mapsto] & \sum \lambda_i x^i \arrow[d, mapsto] \\ & \sum \lambda_i (1\otimes x + x \otimes 1)^i \arrow[d, equal] \\ x \otimes x \arrow[r, mapsto] & (\sum \lambda_i x^i) \otimes (\sum \lambda_j x^j) \end{tikzcd}$$

Assume $\lambda_k \neq 0$ but $\lambda_n = 0 \ \forall n > k$. Then comparing coeff. in front of $x^k \otimes x^k$ yields:

$$\lambda_k^2 = 0 \overset{R \text{ reduced}}{\Longrightarrow} \lambda_k = 0 \ \lightning$$

$\Longrightarrow f(x) = \lambda_0$ for some $\lambda_0 \in R$.

Now consider:

$$\begin{tikzcd} R[x^{\pm1}] \arrow[r, "f"] \arrow[d] & R[x] \arrow[d] \\ R \arrow[r, equal] & R \end{tikzcd} \qquad \begin{tikzcd} x \arrow[r, mapsto] \arrow[d, mapsto] & \lambda_0 \arrow[d, mapsto] \\ 1 \arrow[r, equal] & \lambda_0 \end{tikzcd}$$

$\Longrightarrow f(x) = 1$ and f is given by

$$R[x^{\pm1}] \xrightarrow{e_{\mathbb{G}_m}} R \hookrightarrow R[x]$$

or in terms of group schemes:

$$\mathbb{G}_{a,R} \longrightarrow \underline{1}_R \longrightarrow \mathbb{G}_{m,R}.$$

(3) Assume now that we have $r \in R$ with $r^2 = 0$. Then define

$$R[x] \longrightarrow R[x], \quad x \longmapsto 1 + rx \quad (\text{map of } R\text{-algebras})$$

As $1 + rx \in R[x]^\times$ is a unit, this extends uniquely to a morph. of R-algebras:

$$f: R[x^{\pm1}] \longrightarrow R[x], \quad x \longmapsto 1 + rx, \quad x^{-1} \longmapsto 1 - rx.$$

We claim that f is a morphism of Hopf-algebras:

α) Compatibility with counit:

$$\begin{tikzcd} R[x^{\pm1}] \arrow[r, "f"] \arrow[d, "e"'] & R[x] \arrow[d, "e"] \\ R \arrow[r, equal] & R \end{tikzcd} \qquad \begin{tikzcd} x \arrow[r, mapsto] \arrow[d, mapsto] & 1 + rx \arrow[d, mapsto] \\ 1 \arrow[r, equal] & 1 \end{tikzcd} \qquad \begin{tikzcd} x^{-1} \arrow[r, mapsto] \arrow[d, mapsto] & 1 - rx \arrow[d, mapsto] \\ 1 \arrow[r, equal] & 1 \end{tikzcd}$$

β) Compatibility with comultiplication.

$$\begin{tikzcd} R[x^{\pm1}] \arrow[r, "f"] \arrow[d, "\Delta"'] & R[x] \arrow[d, "\Delta"] \\ R[x^{\pm1}]\otimes R[x^{\pm1}] \arrow[r, "f\otimes f"] & R[x]\otimes R[x] \end{tikzcd} \qquad \begin{tikzcd} x \arrow[r, mapsto] \arrow[d, mapsto] & 1+rx \arrow[d, mapsto] \\ & 1+r(1\otimes x+x\otimes 1) \arrow[d, phantom, "\parallel"] \\ x\otimes x \arrow[r, mapsto] & (1+rx)\otimes(1+rx) \end{tikzcd} \qquad \begin{tikzcd} x^{-1} \arrow[r, mapsto] \arrow[d, mapsto] & 1-rx \arrow[d, mapsto] \\ & 1-r(1\otimes x + x\otimes 1) \arrow[d, phantom, "\parallel"] \\ x^{-1}\otimes x^{-1} \arrow[r, mapsto] & (1-rx)\otimes(1-rx) \end{tikzcd}$$

γ) Compatibility with antipode:

$$\begin{tikzcd} R[x^{\pm1}] \arrow[r, "f"] \arrow[d, "\iota"'] & R[x] \arrow[d, "\iota"] \\ R[x^{\pm1}] \arrow[r, "f"] & R[x] \end{tikzcd} \qquad \begin{tikzcd} x \arrow[r, mapsto] \arrow[d, mapsto] & 1+rx \arrow[d, mapsto] \\ x^{-1} \arrow[r, mapsto] & 1-rx \end{tikzcd} \qquad \begin{tikzcd} x^{-1} \arrow[r, mapsto] \arrow[d, mapsto] & 1-rx \arrow[d, mapsto] \\ x \arrow[r, mapsto] & 1+rx \end{tikzcd}$$

Note that in all diagrams above, all maps are R-algebra morphisms.
Hence it suffices to check commutativity on generators (i.e. on x and x^{-1}).

$\Rightarrow$ $f : R[x^{\pm1}] \longrightarrow R[x]$ is morph. of Hopf-algebras

$\Rightarrow$ f defines morph. between group schemes

$$\mathbb{G}_{a,R} \longrightarrow \mathbb{G}_{m,R}.$$

Ex 4

... was not meant as a mini-project for your TA.

(But if ~~any~~ you have any questions, feel free to ask me nevertheless).