Ex 1) (1) Define $\mathcal{E}^\vee \otimes_{\mathcal{O}_X} \mathcal{F} \longrightarrow \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F})$ on opens U via

$$(\mathcal{E}^\vee \otimes_{\mathcal{O}_X} \mathcal{F})(U) \longrightarrow \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F})(U)$$
$$\sum_i f_i \otimes x_i \longmapsto \left(s \mapsto \sum_i f_i(s) \cdot x_i\right) \quad (f_i \in \mathcal{E}^\vee(U),\ x_i \in \mathcal{F}(U),\ s \in \mathcal{E}(U)).$$

We may check ~~loca~~ bijectivity locally. Hence wlog X affine and $\mathcal{E} \cong \mathcal{O}_X^d$ (for some $d > 0$) and we are reduced to the corresponding statement for modules (use ex. 2!).

(2). Let $f \in \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F}, \mathcal{G})$ and $x \in \mathcal{F}(U)$ (for some open U). Then $f|_U(- \otimes x)$ defines an element in $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{G})(U)$. Thus consider

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F}, \mathcal{G}) \longrightarrow \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{H}om(\mathcal{E}, \mathcal{G}))$$
$$f \longmapsto (x \mapsto f(- \otimes x)).$$

~~Again~~ $\sum s_i \otimes x_i \mapsto \sum \alpha(x_i)(s_i) \longleftarrow\!\shortmid \ \alpha$

Ex 2) We have the map

$$\mathrm{Hom}_{\mathcal{O}_X}(\tilde{M}, \mathcal{F}) \longrightarrow \mathrm{Hom}_R(M, \Gamma(X, \mathcal{F}))$$
$$f \longmapsto f: \tilde{M}(X) = M \to \mathcal{F}(X) \text{ on global sections.}$$

For the converse let $g \in \mathrm{Hom}_R(M, \Gamma(X, \mathcal{F}))$ and define the element in $\mathrm{Hom}_{\mathcal{O}_X}(\tilde{M}, \mathcal{F})$ on the basis given by the standard open $D(f)$ as follows:

$$\tilde{M}(D(f)) = M \otimes_R R[f^{-1}] \xrightarrow{g \otimes id} \mathcal{F}(X) \otimes_R R[f^{-1}] \xrightarrow{\text{rest.}} \mathcal{F}(D(f))$$

Check now that these constructions are mutually inverse.

Ex 3)

Recall that the kernel of a morphism between quasi-coherent sheaves is again quasicoherent. Thus (2) is equivalent to saying that $\mathcal{F}$ admits locally a surjection $\mathcal{O}_U^d \twoheadrightarrow \mathcal{F}|_U$ (for some d depending on U).

As both properties (1) and (2) are local on X it suffices to show:

(1') $\mathcal{F} \cong \tilde{M}$ for some module M over R (where $X = \operatorname{Spec} R$)

(2') $\exists\ \mathcal{O}_X^d \twoheadrightarrow \mathcal{F}$.

For the implication (2') ⇒ (1') use that cokernels of maps between quasi-coherent sheaves are quasi-coherent and for (1') ⇒ (2') use ex. 2 and the statement for modules.

Ex 4) (1) If $x \in U$ then $(j_!\mathcal{F})_x \cong ((j_!\mathcal{F})|_U)_x \cong \mathcal{F}_x$.

If $x \notin U$ then $(j_!\mathcal{F})_x = \varinjlim_V (j_!\mathcal{F})(V)$ is a limit over zero rings!

(We computed both stalks via the presheaf!).

For uniqueness of $j_!\mathcal{F}$ note that any sheaf $\mathcal{G}$ with $\mathcal{G}|_U \cong \mathcal{F}$ we have a canonical morphism of presheaves $(j_!\mathcal{F})^{pre} \to \mathcal{G}$ (with the identity on U).

If $\mathcal{G}_x \cong \mathcal{F}_x$ on U and $\mathcal{G}_x = 0$ else, then it is an isomorphism on all stalks. Hence the corresponding morphism of sheaves is an isomorphism.

(2) Note by uniqueness in (1) we have $(j_!\mathcal{O}_U)|_V \cong (j|_V)_!\,\mathcal{O}_V$. Furthermore $(j|_V)_!\,\mathcal{O}_V \subseteq \mathcal{O}_V$ (as the same is true for $((j|_V)_!\mathcal{O}_V)^{pre}$) and as V is integral we get for any $x \in V \setminus U$:

$$\Gamma(V, (j_!\mathcal{O}_U)|_V) = \Gamma(V, (j|_V)_!\,\mathcal{O}_V) \subseteq \Gamma(V, \mathcal{O}_V) \hookrightarrow \mathcal{O}_{V,x}.$$

But by def. of $j_!\mathcal{O}_U$ the image of every element is 0 in $\mathcal{O}_{V,x}$.

$\Rightarrow \Gamma(V, (j_!\mathcal{O}_U)|_V) = 0$.

But $(j_!\mathcal{O}_U)|_V \neq 0$ as $((j_!\mathcal{O}_U)|_V)|_{U \cap V} \cong \mathcal{O}_{U\cap V} \neq \emptyset$.

Thus $(j_!\mathcal{O}_U)|_V$ is not quasi-coherent, because otherwise (with V affine!):

$$(j_!\mathcal{O}_U)|_V \cong \widetilde{\Gamma(V, j_!\mathcal{O}_U|_V)} = \widetilde{0} = 0 \qquad \text{Contradiction.}$$

(3) Assume $j_!\mathcal{O}_U$ is quasi-coherent. Then choose V as in (2) and it follows $(j_!\mathcal{O}_U)|_V$ is again quasi-coherent.

Contradiction.