Christmas Special

In all exercises K denotes an algebraically closed field and all algebraic groups are defined over this field.

A1 Give an example of an affine variety that cannot occur as the underlying variety of an algebraic group.

A2 The group GL_2 acts naturally on the affine space K^2. Consider the restriction of this action to the subgroup $G = \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \subset GL_2$. What are the orbits of this G-action? Describe the partial order (given by the closure relations) on the set of orbits.

A3 Consider the group $\mathbb{G}_m$ and for any integer $d \geq 0$ the finite subgroup $\mu_d = \{x \in \mathbb{G}_m \,|\, x^d = 1\}$. Compute the quotient $\mathbb{G}_m/\mu_d$.

A4 Compute the Lie algebra $\mathscr{L}(G)$ of the algebraic group $G = SL_n \cap T_n$.

A5 Let G be an algebraic group such that for each representation $\rho : G \to \mathrm{Aut}(V)$ into a finite-dimensional vector space V, there exists a line $K \cdot v \subset V$ fixed under G. Show that G is solvable.

A6 Give an example of a unipotent element in the orthogonal group O_{2014}.

A7 Show that the orthogonal group O_{2015} is not solvable.

A8 Let G_1 and G_2 be two nilpotent algebraic groups. Show that $G_1 \times G_2$ is again nilpotent.

A9 Show that the commutator group (GL_2, GL_2) equals SL_2.

A10 Show that the group of characters $X(GL_2)$ is generated by the character which maps each matrix to its determinant.