Solutions to the Christmas Special

A1 Take $Z = V(xy) \subset K^2$ the union of the two coordinate axis. It is connected, but not irreducible, hence cannot come from any algebraic group.

A2 The orbits are

$$X_y = \{(x,y), x \in K\} \qquad \text{for } y \in K, y \neq 0 \text{ fixed}$$
$$Y = \{(x,0), x \in K \setminus \{0\}\} \quad \text{and} \quad Z = \{(0,0)\}$$

Each X_y and the orbit Z is closed, while the closure of Y is $Y \cup Z$.

A3 Let $\mathbb{G}_m$ act on $X = \mathbb{G}_m$ via $(g,x) = g^d \cdot x$. Then the stabilizer of $1 \in X$ equals μ_d and the orbit of $1 \in X$ is all of X. Hence $\mathbb{G}_m/\mu_d \cong X = \mathbb{G}_m$.

A4 We have $\mathscr{L}(SL_n) = \{X \in M_n \,|\, \mathrm{tr}(X) = 0\}$ and $\mathscr{L}(T_n) = \{X = (x_{ij}) \in M_n \,|\, x_{ij} = 0 \text{ for } i < j\}$. Thus we have

$$\mathscr{L}(G) = \mathscr{L}(SL_n \cap T_n) \subseteq \mathscr{L}(SL_n) \cap \mathscr{L}(T_n) = \{X = (x_{ij}) \in M_n \,|\, \mathrm{tr}(X) = 0 \text{ and } x_{ij} = 0 \text{ for } i < j\}$$

Now $\dim \mathscr{L}(G) = \dim G = \dim T_n - 1$ which equals the dimension of the Lie algebra on the right-hand side. Hence we must have equality.

A5 Choose any faithful representation V of G. Then we may find inductively linearly independent vectors $v_1, \dots, v_n$ such that for each i, the line spanned by v_i inside $V/\langle v_1, \dots, v_{i-1}\rangle$ is fixed by G. Then wrt. to such a basis, G is given by upper triangular matrices in $GL(V)$. Hence it is solvable.

A6 The unit element $1 \in O_{2014}$.

A7 Permutation matrices are orthogonal matrices. Hence O_{2015} contains a closed subgroup isomorphic to the symmetric group S_{2015}. But S_{2015} is not solvable (e.g. because it contains the non-solvable group S_5) and hence O_{2015} cannot be solvable either.

A8 A direct computation gives $\mathcal{C}^i(G_1 \times G_2) = \mathcal{C}^i(G_1) \times \mathcal{C}^i(G_2)$ for each i. Thus for nilpotent G_1 and G_2, we may find some $n \gg 0$ such that $\mathcal{C}^n(G_1) = 1$ and $\mathcal{C}^n(G_2) = 1$. Hence we have found some $n \gg 0$ with $\mathcal{C}^n(G_1 \times G_2) = 1$.

A9 Obviously $(GL_2, GL_2) \subset SL_2$. To show the converse, compute for any $\lambda, \mu, \mu' \in K \setminus \{0\}$:

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \lambda & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ \lambda & 0 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{-1} \in (GL_2, GL_2)$$
$$\begin{pmatrix} 1 & \mu - \mu' \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \mu' & 0 \\ 0 & \mu \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{-1} \cdot \begin{pmatrix} \mu' & 0 \\ 0 & \mu \end{pmatrix}^{-1} \in (GL_2, GL_2)$$
$$\begin{pmatrix} 1 & 0 \\ \mu - \mu' & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} \mu & 0 \\ 0 & \mu' \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^{-1} \cdot \begin{pmatrix} \mu & 0 \\ 0 & \mu' \end{pmatrix}^{-1} \in (GL_2, GL_2)$$

Hence we can decompose

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ a^{-1}c & 1 \end{pmatrix} \cdot \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \cdot \begin{pmatrix} 1 & a^{-1}b \\ 0 & 1 \end{pmatrix} \in (GL_2, GL_2)$$

for any matrix in SL_2 with $a \neq 0$. This shows that (GL_2, GL_2) contains an open subset of SL_2. But (GL_2, GL_2) is closed as well. Hence $(GL_2, GL_2) = SL_2$.

A10 We already saw in one exercise class that any character vanishes on $(GL_2, GL_2) = SL_n$. Hence $X(GL_n) \subset X(GL_n/SL_n) \cong X(\mathbb{G}_m) \cong \mathbb{Z}$. As $X(GL_n)$ is torsion free and non-trivial (because det gives a character), we see $X(GL_2) \cong \mathbb{Z}$. It remains to see that det is indeed a generator of $X(GL_2)$. Assume it is not a generator. Then $\sqrt[n]{\det}$ would be a generator for some $n \geq 2$. But such a morphism

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \sqrt[n]{ad - bc}$$

is not given by a polynomial. Contradiction to our assumption that det is no generator.