Solution to an exercise posed on Monday, 13.10.2014:

We considered

$$\begin{aligned} Ad: GL_n &\longrightarrow GL_{n^2} = GL(Mat(n\times n, K)) \\ A &\longmapsto Ad(A): Mat(n\times n, K) \longrightarrow Mat(n\times n, K) \\ &\qquad\qquad X \longmapsto A^{-1}XA. \end{aligned}$$

a) Convince yourself that this is a morph. of alg. groups.

<u>Sol.</u>
- $Ad(A)$ is a linear map and invertible.
- $A \mapsto Ad(A)$ is a group morphism.
- After fixing a basis of $Mat(n\times n, K)$, we see that Ad is given by polynomials (of degree 2).

b) What is $\ker(Ad)$?

<u>Sol</u>: Any $A \in \ker(Ad)$ must commute with any matrix in $Mat(n\times n, K)$, hence it lies in $Z(GL_n) = \{\lambda \cdot \mathbb{1}_n, \lambda \in K^\times\}$.
In fact we have equality: $Z(GL_n) = \ker(Ad)$.

c) What does this mean for quotients like $G/\ker(Ad|_G)$ if $G \subseteq GL_n$ closed subgroup?

<u>Sol</u>: Just note, that we have defined an algebraic structure on $G/\ker(Ad|_G)$, i.e. $G/\ker(Ad|_G)$ exists again as an alg. group.

d) In the case $SL_2 \xrightarrow{Ad} PSL_2 = \operatorname{im}(Ad) \subseteq GL_4$ and char $K = 2$, show that this is an isomorphism of abstract groups.

<u>Sol</u>: $\ker(Ad|_{SL_2}) = \{\lambda \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in SL_2\} =$

$$= \{\pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in SL_2\} \overset{\text{char}}{\underset{=2!}{=}} \{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\}$$

$\Rightarrow$ kernel is trivial & image is all of PSL_2

$\Rightarrow$ iso of ~~alg.~~ abstract groups.

e) Show that $Ad: SL_2 \longrightarrow PSL_2$ is ~~not~~ an isomorphism of algebraic groups!

Sol: We have to see that Ad is given by polynomials of degree 2. The easiest way (?) to do so is to write down Ad explicitely

$$Ad: SL_2 \longrightarrow PSL_2 \subseteq GL_4$$
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \begin{pmatrix} ad & cd & -ab & -bc \\ bd & d^2 & -b^2 & -bd \\ -ac & -c^2 & +a^2 & ac \\ -bc & -cd & ab & ad \end{pmatrix}$$

Now assume we have an inverse

$$Ad^{-1}: PSL_2 \longrightarrow SL_2$$
$$\begin{pmatrix} \dots \\ \dots \end{pmatrix} \longmapsto \begin{pmatrix} p_1(\dots) & p_2(\dots) \\ p_3(\dots) & p_4(\dots) \end{pmatrix}$$

for some polynomials p_1, p_2, p_3, p_4 in 16 variables (☺).

As $Ad^{-1} \circ Ad = id_{SL_2}$ we compare the upper left component:

$$a = p_1(ad, cd, -ab, -bc, bd, d^2, \dots) \qquad (*)$$
$$\forall a, b, c, d \text{ with } ad - bc - 1 = 0.$$

$\Rightarrow$ (*) holds as equality in $K[a,b,c,d]/(ad-bc-1)$

Now observe that $R = K[a,b,c,d]/(ad-bc-1)$ has a $\mathbb{Z}/2$-grading: $R_0 = \{$all polynomials s.th. all occuring monomials have even degree$\}$

$R_1 = \{$ " odd " $\}$.

The left-hand side of (*) lies in R_1, while the right-hand side lies in R_0. Together with $R_0 \cap R_1 = \{0\}$ we get a contradiction (because $a \neq 0$ in R!).

f)*** Show that $SL_2 \not\cong PSL_2$ (in char. 2) ~~wlog~~

Sol: I have the following two problems

- Unfortunately I couldn't find any proof in the literature
- Straightforward calculations (as we did for endomorphisms of G_m) can be done, but make for a very boring read.

Therefore let me give the following sketch of a proof. ~~The~~ You should be able to understand the basic ideas, but filling in the details might give you problems (at least with your current knowledge): (Assume for simplicity $K = \overline{\mathbb{F}}_2$).

~~Step 1~~ So assume we have an isomorphism $f : SL_2 \longrightarrow PSL_2$

Then all coefficients in all polynomials occurring in the definition of f lie in some field $\mathbb{F}_{2^{m_0}}$ for $m_0 \gg 0$.

Step 1: Show that for all $m \geq m_0$ we have an isomorphism

$$SL_2(\mathbb{F}_{2^m}) \xrightarrow{f} PSL_2(\mathbb{F}_{2^m})$$

(here $SL_2(\mathbb{F}_{2^m}) = SL_2 \cap \mathbb{F}_{2^m}^4 \subseteq K^4 = \overline{\mathbb{F}}_2^4$)

Step 2: Note that Ad and f differ on $SL_2(\mathbb{F}_{2^m})$ by an automorphism of $PSL_2(\mathbb{F}_{2^m})$.

Step 3: People studying finite groups have determined $\mathrm{Aut}(PSL_2(\mathbb{F}_{2^m}))$: It is generated by all inner automorphisms and $\mathrm{Gal}(\mathbb{F}_{2^m}/\mathbb{F}_2)$.

Step 4: Using $SL_2 = \bigcup_m SL_2(\mathbb{F}_{2^m})$ (and similarly for PSL_2) show that we can write (as morphisms of abstract groups!)

$$f = \mathrm{Inn}(A) \circ \sigma \circ \mathrm{Ad} : SL_2 \longrightarrow PSL_2$$

for $\mathrm{Inn}(A)$ the inner automorphism given by conj. by $A \in PSL_2(\overline{\mathbb{F}}_2)$ and $\sigma \in \mathrm{Gal}(\overline{\mathbb{F}}_2/\mathbb{F}_2)$.

Step 5: As $\mathrm{Inn}(A)$ defines an isomorphism of PSL_2 (as algebraic group!), we can assume wlog

$$f = \sigma \circ \mathrm{Ad}. \quad \text{for some } \sigma \in \mathrm{Gal}(\overline{\mathbb{F}}_2/\mathbb{F}_2).$$

Step 6: Show that $\sigma \circ Ad$ is given by polynomials (i.e. is a morphism of alg. groups) if and only if $\sigma = Frob^d$ for some $d \geq 0$.

(Remark: Ad is a morphism of "degree 2" and so is Frob, while isomorphisms should have degree 1. So the main part here is to exclude $Frob^{-1} \circ Ad$ as alg. morphism.).

Step 7: Using a similar argument as we did for Ad, one can show that none of the morphisms $\sigma \circ Ad$ for $\sigma = Frob^d$ is an isomorphism of affine varieties.